Friday, September 01, 2017 11:10 AM

PRIMAL PROBLEM

MAX $u(x_1, x_2)$

s.t. $p_1 x_1 + p_2 x_2 = m$

$\downarrow$

MARSHALLIAN DEMAND FUNCTION

$x_i(p_1, p_2, m)$

$\downarrow$

INDIRECT UTILITY FUNCTION

$v(p_1, p_2, m)$ $\longrightarrow$ EXPENDITURE FUNCTION $e(p_1, p_2, u)$

VIA ROY'S IDENTITY

DUAL PROBLEM

MIN $p_1 x_1 + p_2 x_2 = m$

s.t. $u(x_1, x_2) \geq \bar{u}$

$\downarrow$

HICKSIAN DEMAND FUNCTION

$h_i(p_1, p_2, u)$

VIA SHEPHARD'S LEMMA

$$h_i(p_1, p_2, u) = \frac{\partial e(p_1, p_2, u)}{\partial p_i}$$

MAX $U(x_1, x_2) = x_1^{\alpha} \cdot x_2^{1-\alpha}$

S.T. $p_1 x_1 + p_2 x_2 = m$

LAGRANGIAN FUNCTION:

$$\mathcal{L} = x_1^{\alpha} \cdot x_2^{1-\alpha} + \lambda (m - p_1 x_1 - p_2 x_2)$$

F.O.C:

$$x_1: \frac{\partial \mathcal{L}(\cdot)}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{1-\alpha} - \lambda p_1 = 0 \quad (1)$$

$$x_2: \frac{\partial \mathcal{L}(\cdot)}{\partial x_2} = (1-\alpha) x_2^{-\alpha} x_1^{\alpha} - \lambda p_2 = 0 \quad (2)$$

$$\lambda: \frac{\partial \mathcal{L}(\cdot)}{\partial \lambda} = m - p_1 x_1 - p_2 x_2 = 0 \quad (3)$$

$\frac{(1)}{(2)}$ GIVES

$$\frac{\alpha x_1^{\alpha-1} x_2^{1-\alpha}}{(1-\alpha) x_2^{-\alpha} x_1^{\alpha}} = \frac{\lambda p_1}{\lambda p_2}$$

$$\frac{\alpha}{1-\alpha} \frac{x_2}{x_1} = \frac{p_1}{p_2}$$

$$x_2 = \left(\frac{p_1}{p_2}\right)\left(\frac{1-\alpha}{\alpha}\right) x_1 \quad (4)$$

→ COBB-DOUGLAS UTILITY FUNCTION

Ex: $u(x_1, x_2) = x_1^{\frac{2}{5}} \cdot x_2^{\frac{3}{5}}$

OR

$x_1^{0.4} \cdot x_2^{0.6}$

$\alpha$ = SHARE OF EXPENDITURE ON GOOD 1

$1-\alpha$ = SHARE OF EXPENDITURE ON GOOD 2

$u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$

FIND MRS.

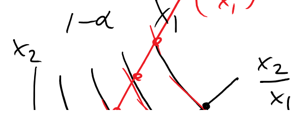
$$MRS_{12} = -\frac{MU_1}{MU_2} = \frac{\frac{du}{dx_1}}{\frac{du}{dx_2}}$$

$$MU_1 = \alpha x_1^{\alpha-1} \cdot x_2^{1-\alpha}$$

$$MU_2 = (1-\alpha) x_2^{1-\alpha-1} x_1^{\alpha} = (1-\alpha) x_1^{\alpha} \cdot x_2^{-\alpha}$$

$$MRS_{12} = -\frac{\alpha x_1^{\alpha-1} x_2^{1-\alpha}}{(1-\alpha) x_1^{\alpha} \cdot x_2^{-\alpha}} = -\frac{\alpha}{1-\alpha} \frac{x_2}{x_1} \quad \text{"}\left(\frac{x_2}{x_1}\right)\text{"}$$

$$\boxed{MRS_{12} = -\frac{\alpha}{1-\alpha} \frac{x_2}{x_1}}$$



$$x_2 = \left(\frac{p_1}{p_2}\right)\left(\frac{1-\alpha}{\alpha}\right) x_1 \quad -④$$

$$④ \rightarrow ③ \quad p_1 x_1 + \cancel{p_2}\left[\left(\frac{p_1}{\cancel{p_2}}\right)\left(\frac{1-\alpha}{\alpha}\right) x_1\right] = m$$

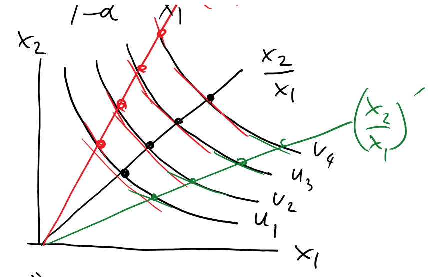
$$p_1 x_1 + p_1\left(\frac{1-\alpha}{\alpha}\right) x_1 = m$$

$$p_1 x_1 \left[1 + \left(\frac{1-\alpha}{\alpha}\right)\right] = m$$

$$(1-\alpha)\, x_1^{\alpha} \cdot x_2^{-\alpha}$$

$$\boxed{MRS'_{12} = -\frac{\alpha}{1-\alpha}\,\frac{x_2}{x_1}}$$

$$|MRS'_{12}| = \frac{\alpha}{1-\alpha}\,\frac{x_2}{x_1}$$




"HOMOTHETICITY"

$$x_1^* = \frac{\alpha m}{p_1} \quad \text{OR} \quad \boxed{x_1^*(p_1, m) = \frac{\alpha m}{p_1}} \quad \text{— (5)}$$

$$(5) \quad \times \quad \left(\frac{1-\alpha}{\cancel{\alpha}}\right)\left[\frac{\cancel{\alpha} m}{\cancel{p_1}}\right]$$

$$x_2 = (1-\alpha)\cdot\frac{m}{p_2} \quad \text{OR} \quad \boxed{x_2^*(p_2, m) = (1-\alpha)\cdot\frac{m}{p_2}} \text{ — (6)}$$

(5) AND (6) ARE SO CALLED "<u>MARSHALLIAN DEMAND FUNCTIONS</u>"

$x_1(p_1, m)$ AND $x_2(p_2, m)$

SUBSTITUTE (5) & (6) TO THE "DIRECT" UTILITY FUNCTION: $U(x_1, x_2)$

$$U(x_1, x_2) = x_1^{\alpha} \cdot x_2^{1-\alpha}$$

$$= \left(\frac{\alpha m}{p_1}\right)^{\alpha}\left(\frac{(1-\alpha)m}{p_2}\right)^{1-\alpha}$$

$$= m^{\alpha} m^{1-\alpha}\left(\frac{\alpha}{p_1}\right)^{\alpha}\left[\frac{(1-\alpha)}{p_2}\right]^{1-\alpha}$$

$$= \underline{\underline{m\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{1-\alpha}{p_2}\right)^{1-\alpha}}}$$

SO, WE NOW OBTAIN "INDIRECT" UTILITY FUNCTION:

$$\boxed{V(p_1, p_2, m) = m\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{1-\alpha}{p_2}\right)^{1-\alpha}} \quad \text{— (7)}$$

<u>SUMMARY:</u>

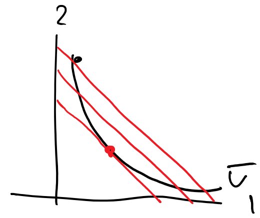
$$\left.\begin{array}{l} \max\ U(x_1, x_2) \\ \text{s.t.} \quad p_1 x_1 + p_2 x_2 = m \end{array}\right\} \xrightarrow{\text{F.O.C}} \begin{array}{l} x_1^*(p_1, m) \\ x_2^*(p_2, m) \end{array} \longrightarrow U(x_1^*, x_2^*)$$

$$\downarrow$$

$V(p_1, p_2, m)$

INDIRECT UTILITY FUNCTION

NEXT, LET'S FIND HICKSIAN DEMAND FUNCTION FROM DUAL PROBLEM.




$$\boxed{\begin{aligned} &\text{MIN} \quad \underline{P_1 X_1 + P_2 X_2} \\ &\text{S.T} \quad \underline{u = x_1^{\alpha} x_2^{1-\alpha}} \end{aligned}}$$

$$\mathcal{L} = \underline{P_1 x_1 + P_2 x_2} + \lambda \left( u - x_1^{\alpha} x_2^{1-\alpha} \right)$$

$$x_1: \quad \frac{\partial \mathcal{L}(\cdot)}{\partial x_1} = P_1 - \lambda \alpha x_1^{\alpha-1} x_2^{1-\alpha} = 0 \quad \text{—} (1)$$

$$x_2: \quad \frac{\partial \mathcal{L}(\cdot)}{\partial x_2} = P_2 - \lambda (1-\alpha) x_2^{-\alpha} x_1^{\alpha} = 0 \quad \text{—} (2)$$

$$\lambda: \quad \frac{\partial \mathcal{L}(\cdot)}{\partial \lambda} = u - x_1^{\alpha} x_2^{1-\alpha} = 0 \quad \text{—} (3)$$

$\frac{(1)}{(2)}$ GIVES

$$\frac{P_1}{P_2} = \frac{\cancel{\lambda} \alpha x_1^{\alpha-1}}{\cancel{\lambda} (1-\alpha) x_1^{\alpha} \quad x_2^{-\alpha}}$$

$$\frac{P_1}{P_2} = \frac{\alpha}{1-\alpha} \frac{x_2}{x_1}$$

$$x_2 = \frac{P_1}{P_2} \left( \frac{1-\alpha}{\alpha} \right) x_1 \quad \text{—} (4)$$

(4) $\rightarrow$ (3) :

$$u = x_1^{\alpha} \left[ \frac{P_1}{P_2} \qquad x_1 \right]^{1-\alpha}$$

$$u = x_1^{\alpha} x_1^{1-\alpha} \left[ \frac{P_1}{P_2} \quad \frac{-\alpha}{\alpha} ) \right]^{1-\alpha}$$

$$u = x_1 \left[ \frac{P_1}{P_2} ( \quad \alpha ) \right]^{1-\alpha}$$

$$x_1 = \left[ \frac{P_2}{P_1} \left( \frac{\alpha}{1-\alpha} \right) \right]^{1-\alpha} \cdot u$$

$$h_1(P_1, P_2, u) = \left( \frac{P_2}{P_1} \right)^{1-\alpha} \left( \frac{\alpha}{1-\alpha} \right)^{1-\alpha} \cdot u \quad \text{—} (5)$$

$$h_1(p_1, p_2, u) = \left(\frac{p_2}{p_1}\right)^{1-\alpha}\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha}\cdot u \quad \text{—} \; (5)$$

"HICKSIAN DEMAND FUNCTION FOR GOOD 1"

$(5) \rightarrow (4)$

$$x_2 = \frac{p_1}{p_2}\left(\frac{1-\alpha}{\alpha}\right)\left[\left(\frac{p_2}{p_1}\right)^{1-\alpha}\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha}\cdot u\right]$$

$$= \left(\frac{p_1}{p_2}\right)\left(\frac{p_2}{p_1}\right)^{1-\alpha}\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha}\cdot u$$

$$= \cancel{\left(\frac{p_1}{p_2}\right)\left(\frac{p_2}{p_1}\right)}\left(\frac{p_2}{p_1}\right)^{-\alpha}\cancel{\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{\alpha}{1-\alpha}\right)}\left(\frac{\alpha}{1-\alpha}\right)^{-\alpha}\cdot u$$

$$= \left(\frac{p_2}{p_1}\right)^{-\alpha}\left(\frac{\alpha}{1-\alpha}\right)^{-\alpha}\cdot u$$

$$x_2 = \left(\frac{p_1}{p_2}\right)^{\alpha}\left(\frac{1-\alpha}{\alpha}\right)^{\alpha}\cdot u$$

; USE MAGIC TRICK $\square^{\Delta} = \frac{1}{\square^{-\Delta}}$

$$h_2(p_1, p_2, u) = \left(\frac{p_1}{p_2}\right)^{\alpha}\left(\frac{1-\alpha}{\alpha}\right)\cdot u$$

HICKSIAN DEMAND FUNCTION FOR GOOD 2 #.

# LET'S DERIVE "EXPENDITURE FUNCTION":

$$e(p_1, p_2, u)$$

HOW? → UTILIZE "INDIRECT UTILITY FUNCTION" TO OBTAIN " EXPENDITURE FUNCTION

$$v(p_1, p_2, m) = m\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{1-\alpha}{p_2}\right)^{1-\alpha}$$

(FROM EQ. 7) FROM PRIMAL PROBLEM

$$\boxed{v(p_1, p_2, e(p_1, p_2, u)) \equiv u}$$

→ DIRECT UTILITY FUNCTION

└ THIS IS AN IDENTITY

$$e(p_1, p_2, u) \cdot \left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{1-\alpha}{p_2}\right)^{1-\alpha} = u$$

SO,
$$e(p_1, p_2, u) = u \cdot \left(\frac{p_1}{\alpha}\right)^{\alpha} \cdot \left(\frac{p_2}{1-\alpha}\right)^{1-\alpha} \quad (6)$$

THIS IS "OUR EXPENDITURE FUNCTION"

# SHEPHARD'S LEMMA:

$$h_i(p_1, p_2, u) = \frac{\partial e(p_1, p_2, u)}{\partial p_i}$$

SO GIVEN
$$e(p_1, p_2, u) = \left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{1-\alpha}\right)^{1-\alpha} u$$
$$= p_1^{\alpha} \cdot \alpha^{-\alpha} \cdot p_2^{1-\alpha} \cdot (1-\alpha)^{\alpha-1} \cdot u$$

$$h_1(p_1, p_2, u) = \frac{\partial e(p_1, p_2, u)}{\partial p_1}$$

$$\boxed{x^{\alpha} = \frac{1}{x^{-\alpha}}}$$

$$= \alpha p_1^{\alpha-1} \cdot p_2^{1-\alpha} \cdot \alpha^{-\alpha} \cdot (1-\alpha)^{\alpha-1} \cdot u$$
$$= p_1^{\alpha-1} p_2^{1-\alpha} \alpha^{1-\alpha} \cdot (1-\alpha)^{\alpha-1} \cdot u$$
$$= \left(\frac{p_2}{p_1}\right)^{1-\alpha}\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} \cdot u \quad \#$$

$$h_2(p_1, p_2, u) = \frac{\partial e(p_1, p_2, u)}{\partial p_2}$$
$$= \left(\frac{p_1}{p_2}\right)^{\alpha}\left(\frac{1-\alpha}{\alpha}\right)^{\alpha} \cdot u \quad \#$$

# ROY'S IDENTITY

$$x_1(p_1, m) = -\frac{\partial v(p_1, p_2, m)}{\partial p_1}$$

$$\boxed{x_1(p_1, m) = -\frac{\dfrac{\partial v(p_1, p_2, m)}{\partial p_1}}{\dfrac{\partial v(p_1, p_2, m)}{\partial m}}}$$

$$\downarrow$$

$$x_1(p_1, m) = \frac{\alpha m}{p_1}$$