Solution: Exercise 1

Part I: Compute the following limits.

1. $\lim\limits_{x\to 25} \frac{5-\sqrt{x}}{25-x}$.
   Solution:

$$\begin{aligned}\lim_{x\to 25}\frac{5-\sqrt{x}}{25-x} &= \lim_{x\to 25}\frac{5-\sqrt{x}}{25-x}\cdot\frac{5+\sqrt{x}}{5+\sqrt{x}} = \lim_{x\to 25}\frac{5^2-(\sqrt{x})^2}{(25-x)(5+\sqrt{x})} = \lim_{x\to 25}\frac{25-x}{(25-x)(5+\sqrt{x})}\\ &= \lim_{x\to 25}\frac{1}{5+\sqrt{x}} = \frac{1}{5+\sqrt{25}} = \frac{1}{10}.\end{aligned}$$

2. $\lim\limits_{x\to 0} \frac{x^2}{\sqrt{x^2+12}-\sqrt{12}}$.
   Solution:

$$\begin{aligned}\lim_{x\to 0}\frac{x^2}{\sqrt{x^2+12}-\sqrt{12}} &= \lim_{x\to 0}\frac{x^2}{\sqrt{x^2+12}-\sqrt{12}}\cdot\frac{\sqrt{x^2+12}+\sqrt{12}}{\sqrt{x^2+12}+\sqrt{12}}\\ &= \lim_{x\to 0}\frac{x^2(\sqrt{x^2+12}+\sqrt{12})}{(\sqrt{x^2+12})^2-(\sqrt{12})^2} = \lim_{x\to 0}\frac{x^2(\sqrt{x^2+12}+\sqrt{12})}{x^2+12-12}\\ &= \lim_{x\to 0}\frac{x^2(\sqrt{x^2+12}+\sqrt{12})}{x^2} = \lim_{x\to 0}\sqrt{x^2+12}+\sqrt{12} = 4\sqrt{3}.\end{aligned}$$

3. $\lim\limits_{x\to 0} \frac{x\sin(x/2)}{|x|}$.
   Solution:

   Note that $|x| = \begin{cases} -x, & \text{for } x<0 \\ x, & \text{for } x>0 \end{cases}$ and $\frac{x}{|x|} = \begin{cases} -1, & \text{for } x<0 \\ 1, & \text{for } x>0 \end{cases}$.
   That is, we can write $-1 \le \frac{x}{|x|} \le 1$. Since $\lim\limits_{x\to 0} -\sin(x/2) = -\sin(0/2) = 0$ and $\lim\limits_{x\to 0}\sin(x/2) = \sin(0/2) = 0$, we can use the *Squeeze Theorem* (Theorem 2.4.1) as follows.

$$\begin{aligned} -1 &\le \frac{x}{|x|} \le 1\\ -\sin(x/2) &\le \frac{x}{|x|}\sin(x/2) \le \sin(x/2)\\ \underbrace{\lim_{x\to 0} -\sin(x/2)}_{=0} &\le \lim_{x\to 0}\frac{x}{|x|}\sin(x/2) \le \underbrace{\lim_{x\to 0}\sin(x/2)}_{=0}\end{aligned}$$

   and therefore, $\lim\limits_{x\to 0}\frac{x}{|x|}\sin(x/2) = 0$ by the Squeeze Theorem.

4. $\lim\limits_{x\to -2} \frac{|x|-2}{\sqrt{x^2-2x}-\sqrt{8}}$.
   Solution:

   Since $|x| = \begin{cases} -x, & \text{for } x<0 \\ x, & \text{for } x>0 \end{cases}$ and since we consider x approaching negative number -2

$(x \to -2)$, we use the condition $|x| = -x$:

$$\begin{aligned}
\lim_{x\to -2} \frac{|x|-2}{\sqrt{x^2-2x}-\sqrt{8}} &= \lim_{x\to -2} \frac{-x-2}{\sqrt{x^2-2x}-\sqrt{8}} \\
&= \lim_{x\to -2} \frac{-x-2}{\sqrt{x^2-2x}-\sqrt{8}} \cdot \frac{\sqrt{x^2-2x}+\sqrt{8}}{\sqrt{x^2-2x}+\sqrt{8}} \\
&= \lim_{x\to -2} \frac{-(x+2)(\sqrt{x^2-2x}+\sqrt{8})}{x^2-2x-8} \\
&= \lim_{x\to -2} \frac{-(x+2)(\sqrt{x^2-2x}+\sqrt{8})}{(x+2)(x-4)} \\
&= \lim_{x\to -2} \frac{-(\sqrt{x^2-2x}+\sqrt{8})}{x-4} \\
&= \frac{-(\sqrt{(-2)^2-2(-2)}+\sqrt{8})}{-2-4} = -\frac{2\sqrt{8}}{-6} = \frac{2\sqrt{2}}{3}.
\end{aligned}$$

5. $\lim_{x\to 2} \frac{|x-2|}{x-2}$.

<u>Solution:</u>

Since $|x-2| = \begin{cases} -(x-2), & \text{for } \ x-2<0 \ \text{ or } \ x<2 \\ x-2, & \text{for } \ x-2>0 \ \text{ or } \ x>2 \end{cases}$,

we have $\frac{|x-2|}{x-2} = \begin{cases} -\frac{(x-2)}{x-2} = -1, & \text{for } \ x<2 \\ \frac{x-2}{x-2} = 1, & \text{for } \ x>2 \end{cases}$. That is, the one-sided limits are

$$\lim_{x\to 2^-} \frac{|x-2|}{x-2} = -1 \qquad \text{and} \qquad \lim_{x\to 2^+} \frac{|x-2|}{x-2} = 1.$$

which are different. Hence, $\lim_{x\to 2} \frac{|x-2|}{x-2}$ does not exist. I.e.,

$$\lim_{x\to 2^-} \frac{|x-2|}{x-2} \neq \lim_{x\to 2^+} \frac{|x-2|}{x-2} \implies \lim_{x\to 2} \frac{|x-2|}{x-2} \quad \text{does not exist.}$$

<u>Part II:</u>

Let $f(x) = \begin{cases} x, & \text{for } \ x<0 \\ x^2, & \text{for } \ 0 \le x < 2 \\ x, & \text{for } \ x>2 \end{cases}$.

1. Find each of the following limits (if exists).

   (a) $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$.

   <u>Solution:</u>

   $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x = 0.$

   $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0^2 = 0.$

(b) $\lim_{x\to 0} f(x)$.
Solution:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 0 \implies \lim_{x\to 0} f(x) = 0.$$

(c) $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$.

Solution:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 = 2^2 = 4.$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x = 2.$$

(d) $\lim_{x\to 2} f(x)$.
Solution:

$$\underbrace{\lim_{x\to 2^-} f(x)}_{=4} \neq \underbrace{\lim_{x\to 2^+} f(x)}_{=2} \implies \lim_{x\to 2} f(x) \quad \text{does not exist.}$$

2. Find all the values of x at which the function $f(x)$ is discontinuous.

Solution:
Since the function $y = x$ and $y = x^2$ are polynomial and hence continuous functions on any given interval, we only have to check at $x = 0$ and $x = 2$.

-For $x = 0$, $f(0)$ is well-defined and $\lim_{x\to 0} f(x)$ exists with $\lim_{x\to 0} f(x) = f(0) = 0$. Then we can conclude that $f(x)$ is continuous at $x = 0$.
-For $x = 2$, since $\lim_{x\to 2} f(x)$ does not exist, then $f(x)$ is discontinuous at $x = 2$.
Therefore the only point at which f is discontinuous is $x = 2$.

Limit and Continuity

1. Find each of the following limits (if it exists).

(a) $\lim_{x\to 8} \frac{x^2-5x-24}{x-8}$
Solution:

$$\lim_{x\to 8} \frac{x^2-5x-24}{x-8} = \lim_{x\to 8} \frac{(x-8)(x+3)}{x-8} = \lim_{x\to 8} x+3 = 11$$

(b) $\lim_{x\to 0} x^3(x^4+2x^3)^{-1}$
Solution:

$$\lim_{x\to 0} \frac{x^3}{x^4+2x^3} = \lim_{x\to 0} \frac{x^3}{x^3(x+2)} = \lim_{x\to 0} \frac{1}{x+2} = \frac{1}{2}$$

(c) $\lim\limits_{x\to 0}\left(\frac{x^3+3x-1}{x}+\frac{1}{x}\right)$
Solution:

$$\lim_{x\to 0}\left(\frac{x^3+3x-1}{x}+\frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{x^3+3x-1+1}{x}\right) = \lim_{x\to 0}\frac{x(x^2+3)}{x} = \lim_{x\to 0} x^2+3 = 3$$

(d) $\lim\limits_{x\to 2}\left(\frac{1}{x-2}-\frac{6}{x^2+2x-8}\right)$
Solution:

$$\begin{aligned}\lim_{x\to 2}\left(\frac{1}{x-2}-\frac{6}{x^2+2x-8}\right) &= \lim_{x\to 2}\left(\frac{1}{x-2}-\frac{6}{(x-2)(x+4)}\right) = \lim_{x\to 2}\left(\frac{x+4-6}{(x-2)(x+4)}\right)\\ &= \lim_{x\to 2}\frac{x-2}{(x-2)(x+4)} = \lim_{x\to 2}\frac{1}{(x+4)} = \frac{1}{6}\end{aligned}$$

(e) $\lim\limits_{x\to 5}\frac{\sqrt{x+4}-3}{x-5}$
Solution:

$$\begin{aligned}\lim_{x\to 5}\frac{\sqrt{x+4}-3}{x-5} &= \lim_{x\to 5}\frac{\sqrt{x+4}-3}{x-5}\cdot\frac{\sqrt{x+4}+3}{\sqrt{x+4}+3} = \lim_{x\to 5}\frac{(\sqrt{x+4})^2-3^2}{(x-5)(\sqrt{x+4}+3)}\\ &= \lim_{x\to 5}\frac{x+4-9}{(x-5)(\sqrt{x+4}+3)} = \lim_{x\to 5}\frac{x-5}{(x-5)(\sqrt{x+4}+3)}\\ &= \lim_{x\to 5}\frac{1}{(\sqrt{x+4}+3)} = \frac{1}{6}\end{aligned}$$

(f) $\lim\limits_{x\to 5}\frac{|x-5|}{x-5}$

Solution: Notice that $|x-5| = \begin{cases} -(x-5), & \text{for } x-5<0 \text{ or } x<5 \\ x-5 & \text{for } x-5>0 \text{ or } x>5 \end{cases}$,

so $\frac{x-5}{|x-5|} = \begin{cases} \frac{(x-5)}{-(x-5)} = -1, & \text{for } x<5 \\ \frac{x-5}{x-5} = 1, & \text{for } x>5 \end{cases}$. That is, the one-sided limits are

$$\lim_{x\to 5^-}\frac{x-5}{|x-5|} = -1 \qquad \lim_{x\to 5^+}\frac{x-5}{|x-5|} = 1,$$

which are different from each other. Hence, $\lim\limits_{x\to 5}\frac{x-5}{|x-5|}$ does not exist. I.e. for $f(x)=\frac{x-5}{|x-5|}$,

$$\lim_{x\to 5} f(x) \quad \text{does not exist because } \lim_{x\to 5^-} f(x) \neq \lim_{x\to 5^+} f(x).$$

(g) $\lim\limits_{x\to 1}\frac{x-1}{\sqrt{5x^2-10x+5}}$
Solution:

$$\begin{aligned}\lim_{x\to 1}\frac{x-1}{\sqrt{5x^2-10x+5}} &= \lim_{x\to 1}\frac{x-1}{\sqrt{5(x^2-2x+1)}} = \lim_{x\to 1}\frac{x-1}{\sqrt{5(x-1)^2}} = \lim_{x\to 1}\frac{x-1}{\sqrt{5}\ \sqrt{(x-1)^2}}\\ &= \lim_{x\to 1}\frac{1}{\sqrt{5}}\frac{x-1}{|x-1|} = \frac{1}{\sqrt{5}}\lim_{x\to 1}\frac{x-1}{|x-1|}\end{aligned}$$

$$|x-1| = \begin{cases} -(x-1), & \text{for} \quad x-1<0 \ \text{ or } \ x<1 \\ x-1 & \text{for} \quad x-1>0 \ \text{ or } \ x>1 \end{cases},$$

so $\frac{x-1}{|x-1|} = \begin{cases} \frac{(x-1)}{-(x-1)} = -1, & \text{for} \quad x<1 \\ \frac{x-1}{x-1} = 1, & \text{for} \quad x>1 \end{cases}.$

That is, for $f(x) = \frac{x-1}{\sqrt{5x^2-10x+5}}$, the one-sided limits are

$\lim_{x\to 1^-} \frac{x-1}{|x-1|} = -1 \implies \lim_{x\to 1^-} f(x) = -\frac{1}{\sqrt{5}}$

$\lim_{x\to 1^+} \frac{x-1}{|x-1|} = 1 \implies \lim_{x\to 1^+} f(x) = \frac{1}{\sqrt{5}}$,

which are not the same. Hence, $\lim_{x\to 5} f(x)$ does not exist. I.e.,

$$\lim_{x\to 1} f(x) \quad \text{does not exist because } \lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x).$$

2. Determine the number (if any) at which each given function is discontinuous.

   (a) $f(x) = x^3 + 2x - 1$

   Solution: None.

   (b) $f(x) = \frac{x^2-1}{x^4-1}$

   Solution: $f(x)$ is discontinuous at $x = \pm 1$ because $x^4 - 1 = 0 \iff x = -1$ or $x = 1$.

   (c) $f(x) = \begin{cases} \frac{x^2-1}{x-1}, & \text{for} \quad x<1 \\ 3, & \text{for} \quad 1 \le x < 2 \\ x^2-1, & \text{for} \quad x \ge 2 \end{cases}$

   Solution: $f(x)$ is discontinuous at $x = 1$ because

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x^2-1}{x-1} = \lim_{x\to 1^-} \frac{(x+1)(x-1)}{x-1} = \lim_{x\to 1^-} x+1 = 1+1 = 2$$

   is different from $\lim_{x\to 1^+} f(x) = 3$.

   Note: $f(x)$ is continuous at $x = 2$ because $\lim_{x\to 2^-} f(x) = 3$ is the same as $\lim_{x\to 2^+} f(x) = 2^2 - 1 = 3$ and so $\lim_{x\to 2} f(x) = f(2) = 3$.

3. Find the values of a, b and c so that the given functions f and g are continuous.

   (a) $f(x) = \begin{cases} ax, & \text{for} \quad x<4 \\ x^2, & \text{for} \quad x \ge 4 \end{cases}$

   Solution: To make $f(x)$ continuous, we only have to make sure that it is continuous at $x = 4$ (since functions $f(x) = ax$ and $f(x) = x^2$ are continuous for all $x \in \mathbb{R}$). Note that

$$\lim_{x\to 4^-} f(x) = 4a \qquad \lim_{x\to 4^+} f(x) = 4^2 = 16 \qquad f(4) = 4^2 = 16.$$

By definition, $f(x)$ is continuous at $x=4$ when $f(4) = \lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x)$. That is,

$$4a = 16 \quad \Longrightarrow \quad a = 4.$$

(b) $g(x) = \begin{cases} bx, & \text{for} \quad x < 3 \\ c, & \text{for} \quad x = 3 \\ -2x+7, & \text{for} \quad x > 3 \end{cases}$

Solution: To make $f(x)$ continuous, we only have to make sure that it is continuous at $x = 3$ (since $g(x) = bx$ and $g(x) = -2x+7$ are continuous for all $x \in \mathbb{R}$). Note that

$$\lim_{x\to 3^-} f(x) = 3b \qquad \lim_{x\to 3^+} f(x) = -2(3)+7 = 1 \qquad f(3) = c.$$

By definition, $f(x)$ is continuous at $x=3$ when $f(3) = \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$. That is,

$$3b = c = 1 \quad \Longrightarrow \quad b = \frac{1}{3}, \quad c = 1.$$

4. **Trigonometric functions:** Find the the following limits.

(a) $\lim_{x\to 0} \frac{x}{\sin(\pi x)}$

Solution: From $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$,

$$\lim_{x\to 0} \frac{x}{\sin(\pi x)} = \lim_{x\to 0} \frac{x}{\sin(\pi x)} \cdot \frac{\pi}{\pi} = \frac{1}{\pi} \lim_{x\to 0} \frac{\pi x}{\sin(\pi x)} = \frac{1}{\pi} \lim_{x\to 0} \frac{1}{\frac{\sin(\pi x)}{\pi x}} = \frac{1}{\pi} \frac{1}{\lim_{x\to 0} \frac{\sin(\pi x)}{\pi x}} = \frac{1}{\pi}.$$

We have used the fact that $\lim_{x\to 0} \frac{\sin(\pi x)}{\pi x} = \lim_{z\to 0} \frac{\sin(z)}{z} = 1$ for $z = \pi x$.

(b) $\lim_{x\to 0} \tan(x)$

Solution: $\lim_{x\to 0} \tan(x) = \tan(0) = 0$.

(c) $\lim_{x\to 0} \frac{\tan(x/2)}{x}$

Solution: From $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$,

$$\begin{aligned} \lim_{x\to 0} \frac{\tan(x/2)}{x} &= \lim_{x\to 0} \frac{\sin(x/2)}{x\cos(x/2)} = \lim_{x\to 0} \left(\frac{1/2}{1/2} \cdot \frac{\sin(x/2)}{x} \cdot \frac{1}{\cos(x/2)} \right) \\ &= \frac{1}{2} \left(\lim_{x\to 0} \frac{\sin(x/2)}{x/2} \right) \left(\lim_{x\to 0} \frac{1}{\cos(x/2)} \right) \\ &= \frac{1}{2} \cdot 1 \cdot \frac{1}{\cos(0/2)} = \frac{1}{2}. \end{aligned}$$

We have used the facts that $\lim_{x\to 0} \frac{\sin(x/2)}{x/2} = \lim_{z\to 0} \frac{\sin(z)}{z} = 1$ for $z = x/2$ and $\cos(0) = 1$.

(d) $\lim_{x\to 2} \frac{\sin(x-2)}{x^2+2x-8}$

Solution: From $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$,

$$\lim_{x\to 2} \frac{\sin(x-2)}{x^2+2x-8} = \lim_{x\to 2} \frac{\sin(x-2)}{(x-2)(x+4)} = \left(\lim_{x\to 2} \frac{\sin(x-2)}{(x-2)}\right)\left(\lim_{x\to 2} \frac{1}{x+4}\right) = 1\cdot\frac{1}{2+4} = \frac{1}{6}.$$

Note that $\lim_{x\to 2} \frac{\sin(x-2)}{x-2} = \lim_{z\to 0} \frac{\sin(z)}{z} = 1$ for $z = x-2$ and $x \to 2 \implies z \to 0$.

(e) $\lim_{x\to 0} \frac{3x+\tan(x)}{x\cos(5x)+\sin(7x)}$

Solution: From $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$,

$$\lim_{x\to 0} \frac{3x+\tan(x)}{x\cos(5x)+\sin(7x)} = \lim_{x\to 0} \frac{[3x+\tan(x)]/x}{[x\cos(5x)+\sin(7x)]/x} = \lim_{x\to 0} \frac{3+\frac{\tan(x)}{x}}{\cos(5x)+\frac{\sin(7x)}{x}} = \frac{3+1}{1+7} = \frac{1}{2},$$

where we have used: $\lim_{x\to 0} \cos(5x) = \cos(0) = 1$,

$$\lim_{x\to 0} \frac{\tan(x)}{x} = \lim_{x\to 0} \frac{\sin(x)}{x\cos(x)} = \left(\lim_{x\to 0} \frac{\sin(x)}{x}\right)\left(\lim_{x\to 0} \frac{1}{\cos(x)}\right) = 1\cdot\frac{1}{\cos(0)} = 1,$$

and

$$\lim_{x\to 0} \frac{\sin(7x)}{x} = \lim_{x\to 0} \frac{\sin(7x)}{x}\cdot\frac{7}{7} = 7\lim_{x\to 0} \frac{\sin(7x)}{7x} = \lim_{z\to 0} \frac{\sin(z)}{z} = 7, \qquad \text{where } z = 7x.$$

(f) $\lim_{x\to 0} x^4 \sin\left(\frac{1}{x}\right)$

Solution: The Squeeze theorem (or the Squeezing/Sandwich theorem) will be used here:

$$\begin{aligned}
-1 \le & \quad \sin\left(\tfrac{1}{x}\right) \quad \le 1 \\
-x^4 \le & \quad x^4\sin\left(\tfrac{1}{x}\right) \quad \le x^4 \\
\underbrace{\lim_{x\to 0}(-x^4)}_{=0} \le & \quad \lim_{x\to 0} x^4\sin\left(\tfrac{1}{x}\right) \quad \le \underbrace{\lim_{x\to 0} x^4}_{=0} \\
0 \le & \quad \lim_{x\to 0} x^4\sin\left(\tfrac{1}{x}\right) \quad \le 0 \qquad \implies \lim_{x\to 0} x^4 \sin\left(\frac{1}{x}\right) = 0.
\end{aligned}$$

That is, by the Squeeze theorem, since
$-x^4 \le x^4\sin\left(\frac{1}{x}\right) \le x^4$, $\lim_{x\to 0} x^4 = 0$ and $\lim_{x\to 0}(-x^4) = -\lim_{x\to 0} x^4 = 0$, then $\lim_{x\to 0} x^4\sin\left(\frac{1}{x}\right) = 0$.

(g) $\lim_{x\to \pi/4} \frac{\tan(x)-1}{\sin(x)-\cos(x)}$

Solution:

$$\begin{aligned}
\lim_{x\to\pi/4} \frac{\tan(x)-1}{\sin(x)-\cos(x)} &= \lim_{x\to\pi/4} \frac{\frac{\sin(x)}{\cos(x)}-1}{\sin(x)-\cos(x)} = \lim_{x\to\pi/4} \frac{\frac{\sin(x)-\cos(x)}{\cos(x)}}{\sin(x)-\cos(x)} \\
&= \lim_{x\to\pi/4} \frac{1}{\cos(x)} = \frac{1}{\cos(\pi/4)} = \frac{1}{1/\sqrt{2}} = \sqrt{2}
\end{aligned}$$

5. **Limit at infinity:** Find the the following limits.

(a) $\lim\limits_{x\to-\infty}\left(\frac{1}{5}x^7-5x^5-x^2\right)$

Solution:

$$\begin{aligned}\lim_{x\to-\infty}\left(\frac{1}{5}x^7-5x^5-x^2\right) &= \lim_{x\to-\infty}x^7\left(\frac{1}{5}-5\frac{x^5}{x^7}-\frac{x^2}{x^7}\right)=\underbrace{\left[\lim_{x\to-\infty}x^7\right]}_{=-\infty}\underbrace{\left[\lim_{x\to-\infty}\left(\frac{1}{5}-\frac{5}{x^2}-\frac{1}{x^5}\right)\right]}_{(1/5-0-0)=1/5}\\ &= -\infty\end{aligned}$$

Note: $\lim\limits_{x\to-\infty}\frac{1}{5}=\frac{1}{5}$, $\lim\limits_{x\to-\infty}\frac{5}{x^2}=0$, and $\lim\limits_{x\to-\infty}\frac{1}{x^5}=0$.

(b) $\lim\limits_{x\to-\infty}\frac{\sqrt{x^2-8}}{1+2x}$ (Note: $\sqrt{x^2}=|x|$)

Solution:

$$\begin{aligned}\lim_{x\to-\infty}\frac{\sqrt{x^2-8}}{1+2x} &= \lim_{x\to-\infty}\frac{\sqrt{x^2(1-\frac{8}{x^2})}}{1+2x}=\lim_{x\to-\infty}\frac{\sqrt{x^2}\ \sqrt{1-\frac{8}{x^2}}}{1+2x}=\lim_{x\to-\infty}\left(\frac{|x|\ \sqrt{1-\frac{8}{x^2}}}{1+2x}\right)\cdot\frac{1/|x|}{1/|x|}\\ &= \lim_{x\to-\infty}\frac{\sqrt{1-\frac{8}{x^2}}}{\frac{1}{|x|}+2\frac{x}{|x|}} = \frac{\sqrt{\lim\limits_{x\to-\infty}1-\lim\limits_{x\to-\infty}\frac{8}{x^2}}}{\lim\limits_{x\to-\infty}\frac{1}{|x|}+2\lim\limits_{x\to-\infty}\frac{x}{|x|}} = \frac{\sqrt{1-0}}{0-2} = -\frac{1}{2}\end{aligned}$$

Note that $\lim\limits_{x\to-\infty}\frac{1}{|x|}=0$ and $\lim\limits_{x\to-\infty}\frac{8}{x^2}=0$. Also, since $|x|=\begin{cases}-x, & \text{for } x<0\\ x & \text{for } x>0\end{cases}$, and therefore

$$x\to-\infty \implies x<0 \implies |x|=-x \implies \frac{x}{|x|}=-\frac{x}{x}=-1 \implies \lim_{x\to-\infty}\frac{x}{|x|}=-1.$$

(c) $\lim\limits_{x\to-\infty}\frac{2x-1}{\sqrt{3x^2-1}+2x}$

Solution:

$$\begin{aligned}\lim_{x\to-\infty}\frac{2x-1}{\sqrt{3x^2-1}+2x} &= \lim_{x\to-\infty}\frac{2x-1}{\sqrt{x^2(3-\frac{1}{x^2})}+2x}=\lim_{x\to-\infty}\frac{2x-1}{|x|\sqrt{(3-\frac{1}{x^2})}+2x}\\ &= \lim_{x\to-\infty}\frac{2\frac{x}{|x|}-\frac{1}{|x|}}{\sqrt{(3-\frac{1}{x^2})}+2\frac{x}{|x|}}=\frac{-2-0}{\sqrt{(3-0)}-2}=\frac{-2}{\sqrt{3}-2}\end{aligned}$$

Note that we have used $\lim\limits_{x\to-\infty}\frac{1}{|x|}=0$, $\lim\limits_{x\to-\infty}\frac{1}{x^2}=0$, and

$$x\to-\infty \implies x<0 \implies |x|=-x \implies \frac{x}{|x|}=-\frac{x}{x}=-1 \implies \lim_{x\to-\infty}\frac{x}{|x|}=-1.$$

(d) $\lim\limits_{x\to\infty}\frac{2x^4-x^2-8x}{1-3x^4}$

Solution:

$$\lim_{x\to\infty} \frac{2x^4 - x^2 - 8x}{1 - 3x^4} = \lim_{x\to\infty} \frac{x^4(2 - \frac{x^2}{x^4} - \frac{8x}{x^4})}{x^4(\frac{1}{x^4} - 3)} = \lim_{x\to\infty} \frac{2 - \frac{1}{x^2} - \frac{8}{x^3}}{\frac{1}{x^4} - 3} = \frac{2 - 0 - 0}{0 - 3} = -\frac{2}{3}$$

Note that we have used $\lim\limits_{x\to\infty} \frac{1}{x^2} = 0$, $\lim\limits_{x\to\infty} \frac{8}{x^3} = 0$ and $\lim\limits_{x\to\infty} \frac{1}{x^4} = 0$.

(e) $\lim\limits_{x\to-\infty} \frac{x^2-8}{1-2x}$

Solution:

$$\lim_{x\to-\infty} \frac{x^2 - 8}{1 - 2x} = \lim_{x\to-\infty} \frac{x(x - \frac{8}{x})}{x(\frac{1}{x} - 2)} = \lim_{x\to-\infty} \frac{(x - \frac{8}{x})}{(\frac{1}{x} - 2)} = \frac{-\infty - 0}{(0 - 2)} = \frac{-\infty}{-2} = \infty$$

Note: Remember that we only look at the denominator when determining the largest power of x here. There is a larger power of x in the numerator but we ignore it. We ONLY look at the denominator when doing this! Otherwise, the denominator could become zero after taking the limit.

(f) $\lim\limits_{x\to\infty} e^{1/x}$

Solution:

$$\lim_{x\to\infty} e^{1/x} = e^{\lim_{x\to\infty} 1/x} = e^0 = 1$$

(g) $\lim\limits_{x\to\infty} \frac{3e^{5x}+2e^{-2x}}{e^{-2x}-e^{5x}}$ and $\lim\limits_{x\to-\infty} \frac{3e^{5x}+2e^{-2x}}{e^{-2x}-e^{5x}}$

Solution:

$$\lim_{x\to\infty} \frac{3e^{5x} + 2e^{-2x}}{e^{-2x} - e^{5x}} = \lim_{x\to\infty} \frac{e^{5x}(3 + 2e^{-7x})}{e^{5x}(e^{-7x} - 1)} = \lim_{x\to\infty} \frac{3 + 2e^{-7x}}{e^{-7x} - 1} = \frac{3 + 0}{0 - 1} = -3$$

$$\lim_{x\to-\infty} \frac{3e^{5x} + 2e^{-2x}}{e^{-2x} - e^{5x}} = \lim_{x\to-\infty} \frac{e^{-2x}(3e^{7x} + 2)}{e^{-2x}(1 - e^{7x})} = \lim_{x\to-\infty} \frac{3e^{7x} + 2}{1 - e^{7x}} = \frac{0 + 2}{1 - 0} = 2$$

Above used the facts that $\lim\limits_{x\to\infty} e^{-7x} = 0$ and $\lim\limits_{x\to-\infty} e^{7x} = 0$.

Note:

- If the limit is at plus infinity we only look at exponentials with positive exponents and if we're looking at a limit at minus infinity we only look at exponentials with negative exponents.
- Remember to only look at the denominator. Do NOT use the exponential from the numerator, even though that one is "larger" than the exponential in the denominator. We always look only at the denominator when determining what term to factor out regardless of what is going on in the numerator.

(h) $\lim\limits_{x\to\infty} (\sqrt{x^2 + 5x} - x)$

Solution: Notice that if ∞ is substituted directly, we obtain the form of $\infty - \infty$. To find the limit, we first *rationalize* the numerator and then multiply both numerator and denominator by $1/|x|$ or $1/x$:

$$
\begin{aligned}
\lim_{x\to\infty}(\sqrt{x^2+5x}-x) &= \lim_{x\to\infty}(\sqrt{x^2+5x}-x)\cdot\frac{(\sqrt{x^2+5x}+x)}{(\sqrt{x^2+5x}+x)} = \lim_{x\to\infty}\frac{(\sqrt{x^2+5x})^2-x^2}{\sqrt{x^2+5x}+x} \\
&= \lim_{x\to\infty}\frac{x^2+5x-x^2}{\sqrt{x^2(1+\frac{5}{x})}+x} = \lim_{x\to\infty}\frac{5x}{\sqrt{x^2}\sqrt{1+\frac{5}{x}}+x} \\
&= \lim_{x\to\infty}\frac{5x}{|x|\sqrt{1+\frac{5}{x}}+x}\cdot\frac{1/x}{1/x} = \lim_{x\to\infty}\frac{5}{\frac{|x|}{x}\sqrt{1+\frac{5}{x}}+1} = \frac{5}{\sqrt{1+0}+1} = \frac{5}{2},
\end{aligned}
$$

where we have used $\sqrt{x^2}=|x|$ and $\frac{|x|}{x}=\frac{x}{x}=1$ for $x>0$ (since $x\to\infty \Rightarrow x>0 \Rightarrow |x|=x$).