Logic

What is logic, and why should we learn about it?

- Logic is used for reasoning.
- Reasoning is the act of deciding if something is true of false.

Learning about logic will enable us to construct better arguments in our essays and papers and to be better at dissecting other people's arguments in our studies.

Section 0.1 Statements

A declarative sentence:
"(noun) is (adjective)"
"(noun) was (adjective)"
"(noun) will be (adjective)"
etc.

- A ***statement*** is a declarative sentence that is either true or false, but not both.

Examples:

(i) "Today is sunny." is a statement.
(ii) "How are you?" is not a statement. (not declarative).
(iii) "School." is not a statement. (not a sentence)
(iv) "1+2 = 3" is a statement.
(v) "This sentence is false." is not a statement. (it is neither true nor false)

- The ***truth value*** of a statement is true or false.

  - It is up to us to decide which statements we will take as true and which ones we will take as false.
  - This will depend on our interpretation of the context of the statement.

- A ***simple statement*** is a statement of the form "(noun) is (adjective)", or other variations.

  - Simple statements can be combined using conjunctive words like "and", "but", "or", "nor", etc. to make ***compound statements***.

Examples:

(i) " It will be sunny tomorrow or it will rain tomorrow."

(ii) "1+3 = 4 and 4+1 = 5."

(iii) "It's not nice outside, but I'm happy."

- A ***conditional statement*** is formed by combining simple statements using "if, then".

e.g. "If it just rained, then the car is wet."

Here, the statement "it just rained" is called the assumption (or hypothesis) and "the car is wet" is called the conclusion.

We can also create quantified statements using quantifiers like "some", "all", and "there exists".

Examples: True or False

(i) "Some dogs are brown."

(ii) "For all integers n, n+1 = 2."

(iii) "There exists a real number x such that $x^2 = 2$."

Section 0.2 Symbolic Logic and Truth Tables

We can use letters, like p and q, to represent statements.

e.g. Let p represent the statement "The sky is blue."

Given two statements, p and q, we write the compound statement "p and q" as

$$\boxed{p \wedge q} \qquad \text{(read: "p and q")}$$

We write the compound statement "p or q" as

$$\boxed{p \vee q} \qquad \text{(read: "p or q")}$$

The statement "It is not true that p" is written as

$$\boxed{\sim p} \qquad \text{(read: "not p")}$$

Examples:

- let p be the statement "Jennie has one brother"
- let q be the statement "Paul has three sisters"
- let r be the statement "Rob has one sister"

(i) $p \wedge q$

(ii) $\sim r \vee q$

(iii) Jennie doesn't have one brother or Paul has three sisters and Rob has one sister.

- <u>The Truth Value of Compound Statements</u>

<u>Examples</u>:

Let

p = "Today is January 12." q = "Today is Monday."

r = "The year is 2012." s = "It is the morning."

True or False:

(i) p (ii) q

(iii) r (iv) s

(v) ~ p (vi) ~ q

(vii) $p \wedge q$ (viii) $p \wedge r$

(ix) $q \vee r$ (x) $q \vee s$

We can summarize the truth values of statements like $p \wedge q$, $p \vee q$, $\sim p$ using truth tables:

<u>AND</u>:

<u>OR</u>:

<u>NOT</u>:

- More Truth Tables

Example: Make a truth table for $(\sim p) \wedge q$.

Example: Make a truth table for $\left[\left(\sim p\right) \wedge q\right] \vee \left(\sim r\right)$

- Always true, always false

A statement which is always true is called a tautology.

Example: $p \vee (\sim p)$ is a tautology:

A statement which is always false is called a contradiction.

Example: $p \wedge (\sim p)$ is a contradiction:

Section 0.3 Conditional Statements

Given two statements, p and q, we write the conditional statement "if p, then q" as

$p \to q$ (read: "p implies q")

Example:

Let p = "It just rained."

and q = "The car is wet."

Then we can write "If it just rained, then the car is wet." as

$p \to q$

Here is the Truth Table for "$\to$" :

IMPLIES:

Other ways to translate $p \to q$:

"p implies q"

"if p, then q"

"if p, q"

"q if p"

"p, therefore q"

"q when p"

Important:

Although $p \wedge q$ is the same as $q \wedge p$ and $p \vee q$ is the same as $q \vee p$, this is not the case with "$\rightarrow$":

$p \rightarrow q$ is not the same as $q \rightarrow p$

In fact, we have a name for $q \rightarrow p$:

We call $q \rightarrow p$ the converse of $p \rightarrow q$.

We see the difference between $p \rightarrow q$ and $q \rightarrow p$ in this truth table:

Another related statement to $p \rightarrow q$ is the inverse:

$$(\sim p) \rightarrow (\sim q)$$

We also have the <u>contrapositive</u> of $p \to q$

$$(\sim q) \to (\sim p)$$

- <u>Biconditional</u>

Biconditional statement is the combination of the statements $p \to q$ and $q \to p$ to form the new statement $(p \to q) \wedge (q \to p)$. This new statement is written as

$$p \leftrightarrow q \qquad \text{(read: “p if and only if q”).}$$

The truth table for $p \leftrightarrow q$:

Section 0.4 Logical Equivalence

Two statements are equivalent if they have the same truth value in every possible situation. The columns of each truth table that were the last to complete will be exactly the same for equivalent statements.

Example: Are the statements $\sim p \vee \sim q$ and $\sim(p \wedge q)$ equivalent?

De Morgan's laws: For any statements p and q

$$\sim(p \vee q) \equiv \sim p \wedge \sim q \quad ; \quad \sim(p \wedge q) \equiv \sim p \vee \sim q$$

Examples: Write the negation of the compound statements.

(i) Colin went to work or he went to bed.

(ii) Jen didn't come last night and didn't pick up her money.

Example: Show that $p \to q$ is equivalent to $\sim p \vee q$

Example: Refer to previous example; use De Morgan's law and double negation to show that $\sim (p \to q) \equiv p \wedge \sim q$

Theorem: For all statements *p*, *q*, and *r*, the following statements are tautologies.

1. Law of Identity:

$$p \leftrightarrow p$$

2. Law of Double Negation:

$$p \leftrightarrow \sim(\sim p)$$

3. Law of Excluded Middle:

$$p \vee \sim p$$

4. Law of Contradiction:

$$\sim(p \wedge \sim p)$$

5. Idempotent Laws:

$$(p \wedge p) \leftrightarrow p$$

$$(p \vee p) \leftrightarrow p$$

6. Law of Addition:

$$p \rightarrow (p \vee q)$$

7. Law of Equivalence:

$$[p \leftrightarrow q] \leftrightarrow [(p \rightarrow q) \wedge (q \rightarrow p)]$$

8. Law of Contraposition:

$$(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$$

9. Law of (Hypothetical) Syllogism:

$$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow [p \rightarrow r]$$

10. Commutative Laws:

$$(p \wedge q) \leftrightarrow (q \wedge p)$$

$$(p \vee q) \leftrightarrow (q \vee p)$$

11. Associative Laws:

$$[p \wedge (q \wedge r)] \leftrightarrow [(p \wedge q) \wedge r] \leftrightarrow [p \wedge q \wedge r]$$

$$[p \vee (q \vee r)] \leftrightarrow [(p \vee q) \vee r] \leftrightarrow [p \vee q \vee r]$$

12. Distributive Laws:

$$\left[p \wedge (q \vee r)\right] \leftrightarrow \left[(p \wedge q) \vee (p \wedge r)\right]$$

$$\left[p \vee (q \wedge r)\right] \leftrightarrow \left[(p \vee q) \wedge (p \vee r)\right]$$

13. Absorption Laws:

$$\left[p \wedge (p \vee q)\right] \leftrightarrow p$$

$$\left[p \vee (p \wedge q)\right] \leftrightarrow p$$

14. De Morgan's Laws:

$$\sim (p \wedge q) \leftrightarrow (\sim p \vee \sim q)$$

$$\sim (p \vee q) \leftrightarrow (\sim p \wedge \sim q)$$

15. Law of Implication:

$$(p \rightarrow q) \leftrightarrow (\sim p \vee q)$$

$$(p \vee q) \leftrightarrow (\sim p \rightarrow q)$$

16. Law of Negation for Implication:

$$\sim (p \rightarrow q) \leftrightarrow (p \wedge \sim q)$$

Example 2: Show that $\left[\sim (\sim p \vee q) \rightarrow p\right]$ is tautology.

Example 3: Show that $(p \to q) \wedge p \wedge \sim q$ is contradiction.