

EE320 (1/2015)

INTRODUCTORY MATHEMATICAL ECONOMICS

NONLINEAR MODEL AND DIFFERENTIAL CALCULUS
IN ECONOMIC THEORY

Topics

- Slopes of Curves: Linear and Non-linear Models
- Slope and Derivatives of a Function
- Differentiability of a Function
- Rules of Differentiation
- Examples in Economics:
 - Derivative and Marginality
 - Relations among the total, the average, and the marginal functions

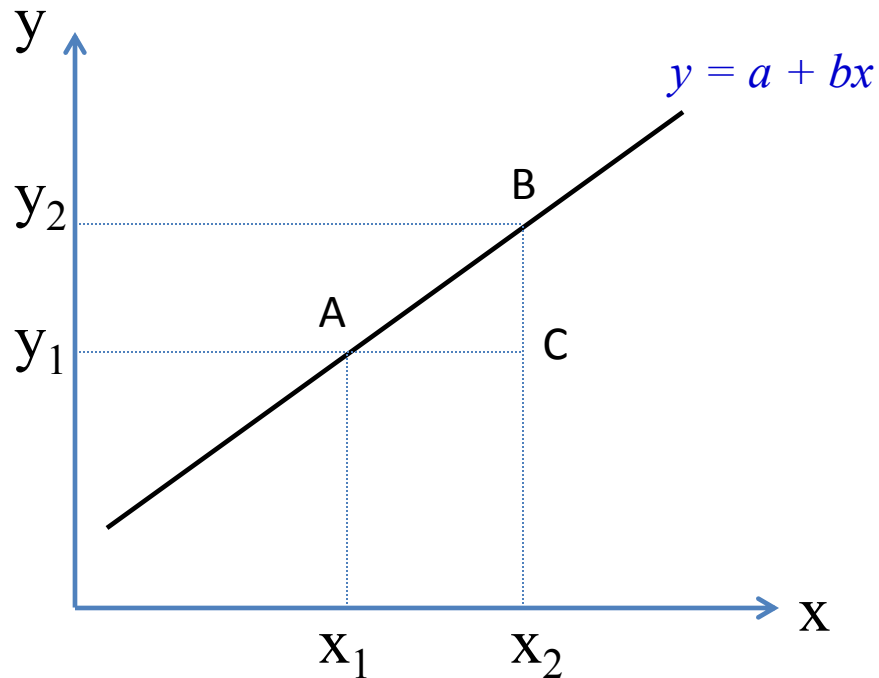
REVIEW DIFFERENTIAL CALCULUS

Introduction

- This lecture focuses *the rate of change* of the equilibrium value of an endogenous variable (y) with respect to the change in a particular exogenous variable (x), given $y = f(x)$.
- Geometrically, this rate of change is referred to as the *slope of a curve*.
- **Examples:**
 - Slope of a *total cost* function is the *marginal cost*.
 - Slope of a *utility* function is *marginal utility*.
- The slope of a function tells the characteristics of the function.
 - **Linear functions:** Slope is constant.
 - **Nonlinear functions:** Slopes vary at different values of x .

Linear Function

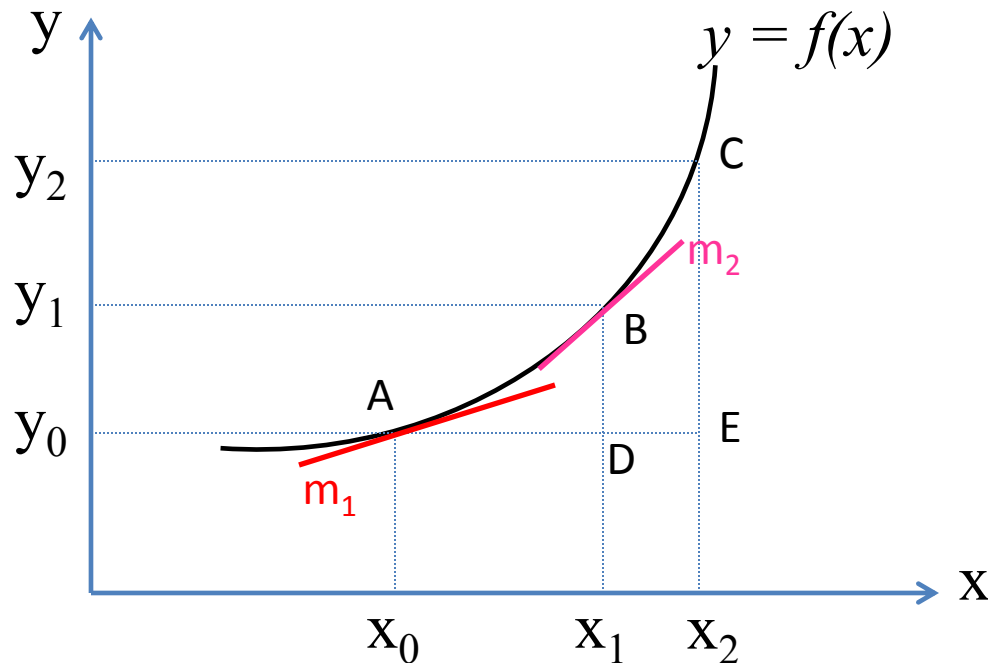
- Consider a linear function: $y = a + bx$.
- Graph



- Slope between point A and B is: $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{BC}{AC}$

Nonlinear Function

- Consider the following non-linear function



- The slope between any two points vary along the curve.
- The slope of the curve must be measured at the tangency of a particular point.

Rate of Change and the Derivative (1)

- Given the function $y = f(x)$, the **difference quotient** is the change in y per unit of change in x :

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

➤ i.e. the difference quotient measures the average rate of change of y .

- Example: Given $y = f(x) = x^2 + 3x$, we can write:

$$f(x_0) = x_0^2 + 3x_0$$

$$f(x_0 + \Delta x) = (x_0 + \Delta x)^2 + 3(x_0 + \Delta x)$$

➔
$$\frac{\Delta y}{\Delta x} = 2x_0 + \Delta x + 3$$

Rate of Change and the Derivative (2)

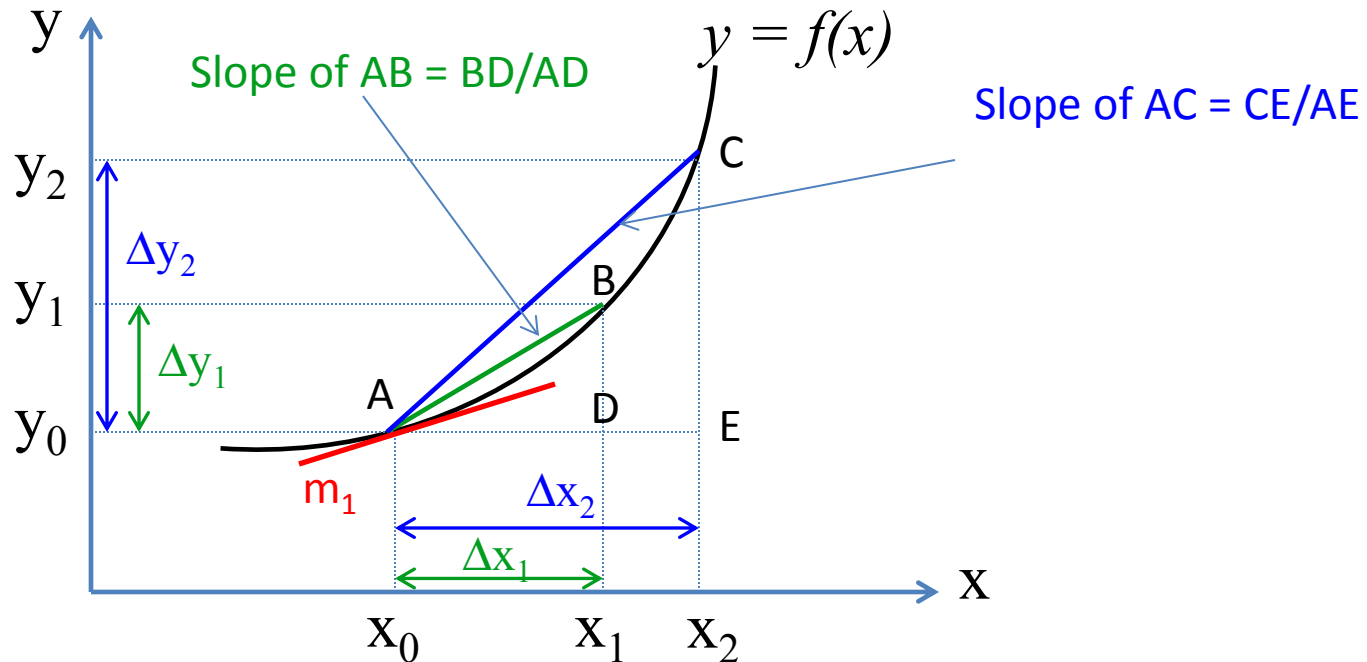
- When Δx is *infinitesimally small*, the rate of change of y is called the **derivative**:

$$\frac{dy}{dx} \equiv f'(x) \equiv \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

- i.e. the derivative of the function $y = f(x)$ is the **limit of the difference quotient** $\Delta y/\Delta x$.
- The derivative measures the **instantaneous rate of change**.
- Example: Given $y = f(x) = x^2 + 3x$,

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} 2x_0 + \Delta x + 3 = 2x_0 + 3$$

Slope and Derivatives



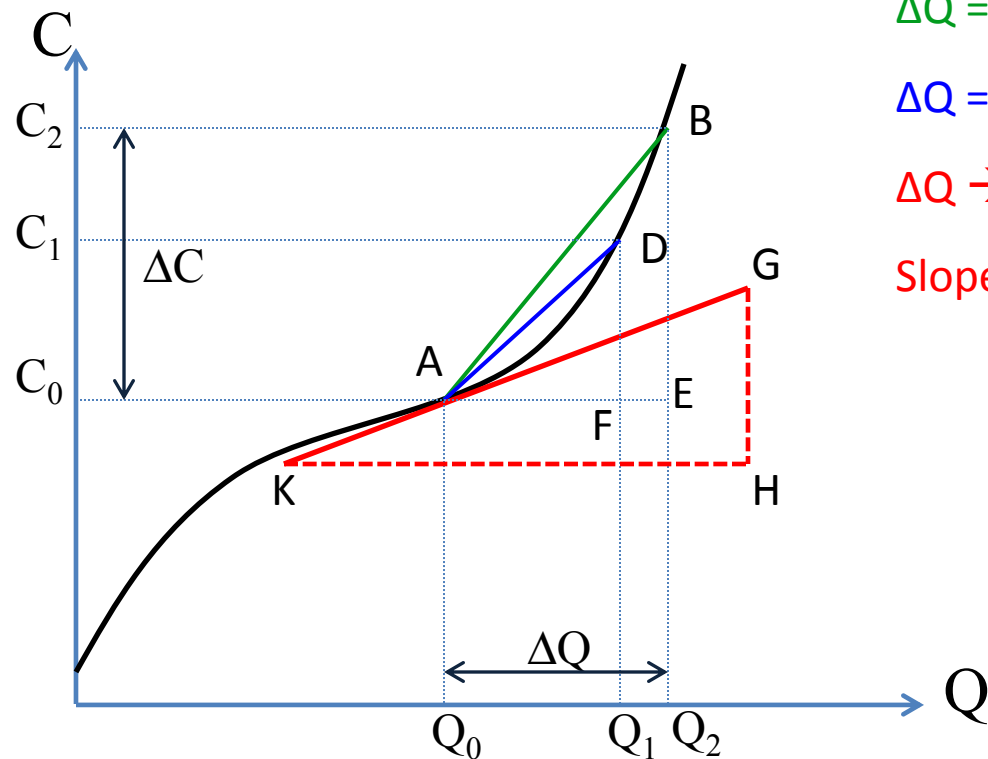
Slope at point A = Slope of m_1

$$= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

The tangency to the curve $y = f(x)$ at $x=a$ is the slope: $m = f'(a)$.

Slope and Derivatives

- **Example:** Total cost curve and its slope



$\Delta Q = Q_2 - Q_0$: Slope of AB = BE/AE

$\Delta Q = Q_1 - Q_0$: Slope of AD = DF/AF

$\Delta Q \rightarrow 0$:

Slope at A = Slope of KG (tangent at A)
 $= GH/KH$
 $= \lim_{\Delta Q \rightarrow 0} (\Delta C / \Delta Q)$
 $= f'(Q_0)$

Differentiability of a Function

- The process of determining the derivative of a function is called “differentiation”
- A function is differentiable if it is continuous and smooth.
 - Continuity is a *necessary condition* for differentiability.
 - Smoothness is a *sufficient condition* for differentiability.

- Definition:

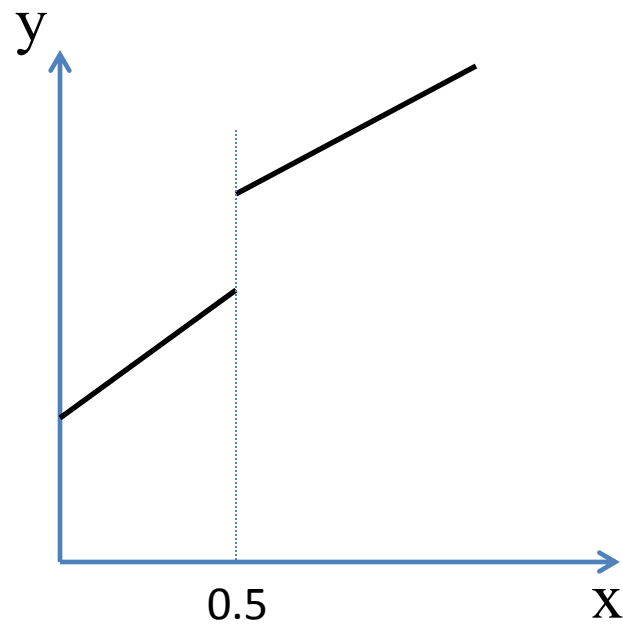
A function $y = f(x)$ is continuous at x_0 if the followings are true:

1. $f(x_0)$ is defined.
2. $\lim_{x \rightarrow x_0}$ exists.
3. $\lim_{x \rightarrow x_0} f(x) = f(x_0)$

Continuity

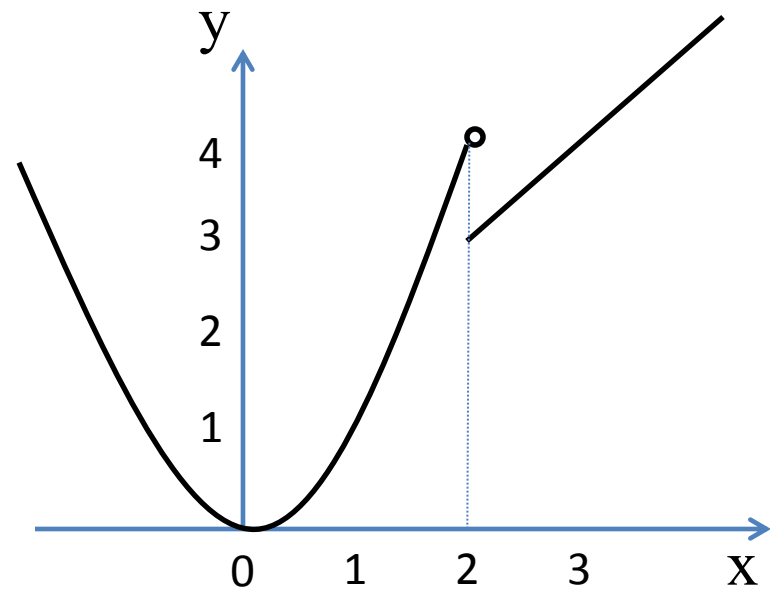
- Example 1:

$f(x)$ is discontinuous at 0.5.



- Example 2:

$$g(x) = \begin{cases} x^2 & \text{if } x < 2 \\ x+1 & \text{if } x \geq 2 \end{cases}$$



Smoothness

- A function is **smooth** means that it is differentiable everywhere.
- Definition:

A function $f(x)$ is differentiable at point a if

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

Alternatively, given that $x = a + h$, then $x \rightarrow a$ if $h \rightarrow 0$. Then,

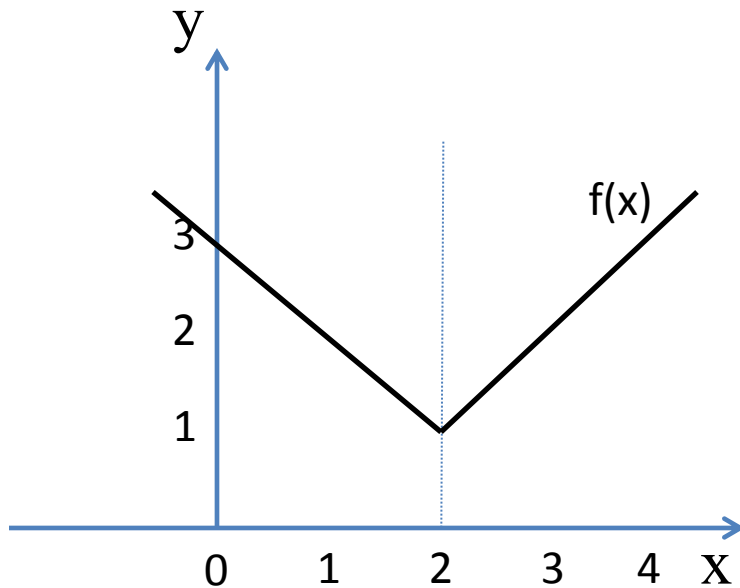
$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}, x \neq a$$

- Theorem:

If a function $f(x)$ has a derivative at a , then it is continuous at a as well.

Non-Differentiable Functions

- Example: $f(x) = |x - 2| + 1$



$$\lim_{x \rightarrow 2^-} \frac{f(x) - f(2)}{x - 2} = -1$$

$$\lim_{x \rightarrow 2^+} \frac{f(x) - f(2)}{x - 2} = 1$$

$$\therefore \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} \text{ does not exist.}$$

Thus, $f(x)$ is not differentiable at $x = 2$.

Summary: All differentiable functions are continuous, but not all continuous functions are differentiable.

Rules of Differentiation:

A Function of One Variable

- Constant-Function Rule:

$$y = f(x) = c, \text{ where } c \text{ is a constant}$$

$$\rightarrow f'(x) = 0.$$

- Power-Function Rule:

$$y = f(x) = x^n, \text{ where } n \text{ is a real number}$$

$$\rightarrow f'(x) = nx^{n-1}$$

- Exponential-Function Rule:

$$y = f(x) = e^x$$

$$\rightarrow f'(x) = e^x$$

- Logarithm-Function Rule:

$$y = f(x) = \ln(x)$$

$$\rightarrow f'(x) = 1/x$$

Rules of Differentiation:

Two or More Functions of the Same Variable

- Sum-Difference Rule:

$$y = f(x) + g(x) \rightarrow \frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

$$y = f(x) - g(x) \rightarrow \frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

- Product Rule:

$$y = f(x)g(x) \rightarrow \frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

- Quotient rule:

$$y = \frac{f(x)}{g(x)}$$

$$\rightarrow \frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Examples

- Example 1:

$$f(x) = 2x^3 + 3x^2 - 5x + 1$$

$$f'(x) = 6x^2 + 6x - 5$$

- Example 2:

$$f(x) = 3x^{-2/3}$$

$$f'(x) = -2x^{-5/3}$$

- Example 3:

$$f(t) = e^{rt}$$

$$f'(t) = re^{rt}$$

Examples

- Example 4:

$$f(x) = (6x + 1)(x^2 - 2x)$$

$$f'(x) = 18x^2 - 22x - 2$$

- Example 5:

$$f(x) = (x + 1)/2x^2$$

$$f'(x) = -(x + 2)/2x^3, \quad \text{for } x \neq 0$$

- Example 6:

$$f(x) = x^5(3x)^2$$

$$f'(x) = 63x^6$$

Rules of Differentiation:

Functions of Different Variables

- Chain Rule:

$$z = f(y) \text{ and } y = g(x) \rightarrow z = f(g(x))$$

$$\rightarrow \frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx} = f'(y)g'(x)$$

- Inverse-Function Rule:

If the function $y = f(x)$ represents a one-to-one mapping, the function f will have an inverse function: $x = f^{-1}(y)$.

$$\rightarrow \frac{dx}{dy} = \frac{1}{dy/dx}$$

Examples

- Example 7:

$$f(x) = (x^2 - 4x)^3$$

$$f'(x) = 6x^5 - 60x^4 + 192x^3 - 192x^2$$

- Example 8:

$$f(x) = \ln(2x^2)$$

$$f'(x) = 2/x$$

- Example 9:

$$f(x) = 3x^5 + x^2$$

$$dx/dy = 1/(15x^4 + 2x)$$

APPLICATIONS IN ECONOMICS

Derivative and Marginality

- In economics, *marginality* indicates a rate of change – how much *the value of the function* changes as the *choice variable* (independent variable) increases by one unit.
- Examples:

	Level	Average	Marginal
Production	$TP=Q = g(L)$	$AP = TP/L$	$MP = d(TP)/dL$
Revenue	$TR = f(Q)$	$AR = TR/Q$	$MR = d(TR)/dQ$ $MRP = d(TR)/dL$
Cost	$TC = h(Q)$	$AC = TC/Q$	$MC = d(TC)/dQ$ $MFC = d(TC)/dL$
Consumption	C	APC	MPC
Saving	S	APS	MPS
Investment	I		MPI

Total, Average, and Marginal Revenue

- Total Revenue: $TR = P \times Q$.
- Demand function: $Q_d = a - bP$.
 - Example: $Q_d = 250 - 10P$. $\rightarrow P = 25 - 0.1Q$
- $TR = f(Q) = P(Q) \times Q = (25 - 0.1Q) \times Q$
 $\rightarrow TR(Q) = 25Q - 0.1Q^2$
- $AR = TR/Q = 25Q - 0.1Q \rightarrow AR(Q) = P(Q)$
- $MR = d(TR)/dQ = 25Q - 0.2Q$
- ❖ From $TR = P(Q) \times Q$ and $MR = d(TR)/dQ$,
 we have $MR = \frac{d}{dQ}[P(Q) \times Q] = P(Q) \cdot \frac{dQ}{dQ} + Q \cdot \frac{dP}{dQ}$
 Thus, $MR = AR + Q \cdot \frac{dP}{dQ}$, $\because P(Q) = AR$

Elasticity, Total Revenue, and Marginal Revenue

- From above, $MR = AR + Q \cdot \frac{dP}{dQ}$

$$MR = AR + Q \cdot \frac{dP}{dQ} \cdot \frac{P}{P}$$

$$MR = AR + \frac{P}{\frac{dQ/Q}{dP/P}} = AR + \frac{P}{\epsilon_d} = AR + \frac{AR}{\epsilon_d}$$

$$\therefore MR = AR \left(1 + \frac{1}{\epsilon_d} \right)$$

- If $|E_d| > 1$ (elastic), $MR > 0$
- If $|E_d| < 1$ (inelastic), $MR < 0$
- If $|E_d| = 1$ (unitary elastic), $MR = 0$

Example: Elasticity, MR, and AR

- Suppose $TR = 25Q - 0.1Q^2$

[See lecture notes.]

Total, Average, and Marginal Product

- Consider a short-run production: $Q = g(L)$

- $AP = Q/L = g(L)/L$

- $MP = dQ/dL = d[g(L)]/dL = g'(L)$

- Relation between AP and MP:

$$\frac{d(AP)}{dL} = \frac{d}{dL} \left[\frac{g(L)}{L} \right] = \frac{L \cdot g'(L) - g(L)(1)}{L^2} = \frac{1}{L} \left[g'(L) - \frac{g(L)}{L} \right] = \frac{1}{L} [MP(L) - AP(L)]$$

- Marginal Revenue Product: **MRP**

$$MRP = \frac{d}{dL} [P(Q(L)) \times Q(L)]$$

[See additional notes for the derivation.]

From above, we can show that $MRP = MP(L) \times MR(Q)$.

Total, Average, and Marginal Cost

- $TC = FC + VC = a + f(Q) = C(Q)$

- $AC = C(Q)/Q$

- $MC = d(TC)/dQ = C'(Q)$

- Relation between AC and MC:

$$\frac{d(AC)}{dQ} = \frac{d}{dQ} \left[\frac{C(Q)}{Q} \right] = \frac{Q \cdot C'(Q) - C(Q)(1)}{Q^2} = \frac{1}{Q} \left[C'(Q) - \frac{C(Q)}{Q} \right] = \frac{1}{Q} [MC(Q) - AC(Q)]$$

- Marginal Factor Cost: **MFC**

$$MFC = \frac{d}{dL} [C(Q(L))] = \frac{dC}{dQ} \cdot \frac{dQ}{dL} = MC(Q) \times MP(L)$$