(a)

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \to \begin{bmatrix} 3 & -1 & 0 \\ 0 & 8/3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \to \begin{bmatrix} 3 & -1 & 0 \\ 0 & 8/3 & -1 \\ 0 & 0 & 21/8 \end{bmatrix}.$$

Thus $\det(A) = 3(8/3)(21/8) = 21$.

(b) Since $\det(A)$ is nonzero, A is invertible. $\det(A^{-1}) = (\det(A))^{-1} = 1/21$.

(c)

$$\det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & -1 & 0 \\ -1 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{bmatrix}$$

$$= (3-\lambda)\det\begin{bmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{bmatrix} - (-1)\det\begin{bmatrix} -1 & -1 \\ 0 & 3-\lambda \end{bmatrix} + 0 = (3-\lambda)^3 - 2(3-\lambda$$

2. a) $-24+26t+3t^2-6t^3+t^4$

b) -24

c) -384

d) 0

e) $-8, 1, 27, 64$

f) $-8 \times 1 \times 27 \times 64$

g) yes

h)

3) a)

$$A = PDP^{-1} = \begin{bmatrix} 1 & 6 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -6 & -8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{\infty} = \begin{bmatrix} 1 & 6 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -6 & -8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. a)

$$A = \begin{bmatrix} 7 & 5 \\ 3 & -7 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\lambda^2 - 64 = 0$$

$$\lambda = \pm 8$$

For $\lambda_1 = 8$, the corresponding eigenvector is

$$(A - 8I)v_1 = 0$$

$$v_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

For $\lambda_2 = -8$, the eigenvector is

$$(A + 8I)v_2 = 0$$

$$v_2 = \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$$

$$\therefore P = \begin{bmatrix} 5 & -1/3 \\ 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 8 & 0 \\ 0 & -8 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} \frac{3}{16} & \frac{1}{16} \\ \frac{-3}{16} & \frac{15}{16} \end{bmatrix}$$

b) $B^3 = A$

$B = A^{1/3}$

$$A = PDP^{-1} \; ; \quad A^{1/3} = PD^{1/3}P^{-1} = \frac{1}{4}\begin{bmatrix} 7 & 5 \\ 3 & -7 \end{bmatrix}.$$

(3). $A = PDP^{-1}$

$P = \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$; $D = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$

$\therefore P^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ -1 & 2 \end{bmatrix}$

$\therefore A = \begin{bmatrix} -9 & 14 \\ -7 & 12 \end{bmatrix}$

6). a) $n - r$ = dimension of the null space of A = the number of eigen values of A which are 0.

So rank of $A = 2$

b) $\det(A^T A) = \det(A^T) \cdot \det(A)$

$\det A = \lambda_1 \lambda_2 \lambda_3 = 0$

$\therefore \det(A^T A) = \det(A^T)\det(A) = 0$

c) $\det(A + I) = 1 \times 2 \times 3 = 6$

(7) $\begin{vmatrix} 1+a & b & c & d \\ a & 1+b & c & d \\ a & b & 1+c & d \\ a & b & c & 1+d \end{vmatrix} \underset{R_4 - R_1}{\overset{R_2 - R_1,\ R_3 - R_1}{\sim}} \begin{vmatrix} 1+a & b & c & d \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{vmatrix}$

Use column operations one operations at a time

$C_1 + C_2$ then $C_1 + C_3 \to C_1 + C_4$ (to eliminate ⊖ underneath pivot)

$\begin{vmatrix} 1+a+b+c+d & b & c & d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \underline{(1+a+b+c+d)}$

Exercise VII

1.

$\det(A-\lambda I) = 0$

$\lambda^2 - (a+c)\lambda + ac - b^2 = 0$

$$\lambda_{1,2} = \left(\frac{(a+c) \pm \sqrt{(a-c)^2 + 4b^2}}{2} \right)$$

$\lambda_1 > 0$ is obvious

$\lambda_2 > 0$ because $(a+c)^2 > (a-c)^2 + 4b^2$ is equivalent to $ac > b^2$

2).

$a > 1$

There is no b that makes B positive definite.

3. Let $A = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 2 \end{bmatrix}$ one may compute that

$$A \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -20 \\ 20 \\ 10 \end{bmatrix} = 10 \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

So $v_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvector of A with associated eigenvalue

$\lambda_1 = 10$. Likewise one may compute that

$$A \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

So $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector of A with associated eigenvalue $\lambda_2 = 1$. For $\lambda_2 = 1$, one computes that a basis for