Solution: Quiz 5

1. Define $H : \mathbb{Z}^+ \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ as follows:

$$H(a, b) = (3a^2 - 1, 4b + 1) \text{ for all } (a, b) \in \mathbb{Z}^+ \times \mathbb{Z},$$

where $\mathbb{Z}^+$ is the set of all positive integers and $\mathbb{Z}$ is the set of all integers.

(a) Is H one-to-one? Prove or give a counterexample.

(b) Is H onto? Prove or give a counterexample.

(c) Is H bijective? If so, find H^{-1}, the inverse function of H.

Solution:

(a) Is H one-to-one? Prove or give a counterexample.
Answer: Yes, H is one-to-one. Let $(a_1, b_1), (a_2, b_2)$ be some elements in the domain $\mathbb{Z}^+ \times \mathbb{Z}$. We want to show that if $H(a_1, b_1) = H(a_2, b_2)$, then $(a_1, b_1) = (a_2, b_2)$.
Suppose $H(a_1, b_1) = H(a_2, b_2)$. Then

$$(3a_1^2 - 1, 4b_1 + 1) = (3a_2^2 - 1, 4b_2 + 1)$$

or equivalently, $3a_1^2 - 1 = 3a_2^2 - 1$ and $4b_1 + 1 = 4b_2 + 1$. I.e.,

$$3a_1^2 - 1 = 3a_2^2 - 1 \quad \Leftrightarrow \quad 3a_1^2 = 3a_2^2 \quad \Leftrightarrow \quad a_1^2 = a_2^2 \quad \Leftrightarrow \quad a_1 = \pm a_2 \quad \Leftrightarrow \quad a_1 = a_2,$$

(where we have used the fact that both a_1 and a_2 are in $\mathbb{Z}^+$ and so $a_1^2 = a_2^2 \Leftrightarrow a_1 = a_2$ for $a_1, a_2 > 0$), and

$$4b_1 + 1 = 4b_2 + 1 \quad \Leftrightarrow \quad 4b_1 = 4b_2 \quad \Leftrightarrow \quad b_1 = b_2.$$

That is, $H(a_1, b_1) = H(a_2, b_2)$ implies $(a_1, b_1) = (a_2, b_2)$ and therefore, H is one-to-one.

(b) Is H onto? Prove or give a counterexample.
Answer: No, H is not onto. Notice that if we pick (u, v) from the co-domain $\mathbb{Z} \times \mathbb{Z}$ and suppose that there is (a, b) in the domain such that $H(a, b) = (u, v)$, then

$$(u, v) = H(a, b) = (3a^2 - 1, 4b + 1)$$

or we must have

$$u = 3a^2 - 1 \quad \Rightarrow \quad a = \pm\sqrt{\frac{u + 1}{3}}$$

and

$$v = 4b + 1 \quad \Rightarrow \quad b = \frac{v - 1}{4}.$$

So, the following is a counterexample for this. If we choose $(u, v) = (0, 0)$ from the co-domain $\mathbb{Z} \times \mathbb{Z}$, then, in order to have $H(a, b) = (0, 0)$, we must set $a = \sqrt{\frac{0+1}{3}} = \sqrt{\frac{1}{3}}$ or $a = -\sqrt{\frac{0+1}{3}} = -\sqrt{\frac{1}{3}}$, and set $b = \frac{0-1}{4} = \frac{-1}{4}$. However, $a = \pm\sqrt{\frac{1}{3}}, b = -\frac{1}{4}$ are not in $\mathbb{Z}$ and therefore, we **cannot** find any element (a, b) from the domain $\mathbb{Z}^+ \times \mathbb{Z}$ such that $H(a, b) = (0, 0)$. That is, H is not onto.

(c) Is H bijective? If so, find H^{-1}, the inverse function of H.
Answer: No, H is not bijective because it is nor onto. So we cannot find its inverse function.