Solution: Quiz 4

1. Find the solution set for
$$\left|\frac{1-2x}{x^2+1}\right| \leq \frac{2x-1}{2}.$$

Answer: Notice that
$$|1-2x| = \begin{cases} -(1-2x), & \text{for} \quad 1-2x<0, \text{ or } x>1/2 \\ 1-2x, & \text{for} \quad 1-2x \geq 0, \text{ or } x \leq 1/2 \end{cases}$$
Notice also that $x^2 \geq 0$ and hence $x^2+1>0$. So, we have $|x^2+1| = x^2+1$. That is, $\left|\frac{1-2x}{x^2+1}\right| = \frac{|1-2x|}{|x^2+1|} = \frac{|1-2x|}{x^2+1}$
$$\frac{|1-2x|}{x^2+1} \leq \frac{2x-1}{2}.$$
We will consider 3 cases for values of x: (I) $x=1/2$ (II) $x \in (-\infty, 1/2)$ and (III) $x \in (1/2, \infty)$

Case I: $x = 1/2$. We have that the inequality is true because $\left|\frac{0}{0^2+1}\right| \leq \frac{0}{2}$. That is, the solution set is $\{\frac{1}{2}\}$.

Case II: $x \in (-\infty, 1/2)$ or $x < 1/2$. In this case $|1-2x| = 1-2x$, and we have
$$\frac{|1-2x|}{x^2+1} \leq \frac{2x-1}{2} \quad \text{or} \quad \frac{2}{x^2+1} \leq \frac{2x-1}{|1-2x|} \quad \text{or} \quad \frac{2}{x^2+1} \leq \frac{2x-1}{1-2x} = -1,$$
$$\frac{2}{x^2+1} + 1 \leq 0$$
which is always false for $x \in \mathbb{R}$ because $x^2+1>0 \Rightarrow \frac{2}{x^2+1} > 0$ and therefore $1 + \frac{2}{x^2+1} > 0$. Hence, the solution set is $(-\infty, 1/2) \cap \emptyset = \emptyset$.

Case III: $x \in (1/2, \infty)$ or $x > 1/2$. In this case $|1-2x| = -(1-2x) = -1+2x = 2x-1$, and we have
$$\frac{|1-2x|}{x^2+1} \leq \frac{2x-1}{2} \quad \text{or} \quad \frac{2}{x^2+1} \leq \frac{2x-1}{|1-2x|} \quad \text{or} \quad \frac{2}{x^2+1} \leq \frac{2x-1}{2x-1},$$
$$\frac{2}{x^2+1} \leq 1$$
$$1 - \frac{2}{x^2+1} \geq 0$$
$$\frac{x^2+1-2}{x^2+1} \geq 0 \quad \text{or} \quad \frac{x^2-1}{x^2+1} \geq 0$$
which happens if and only if $x^2-1 \geq 0$ or $(x+1)(x-1) \geq 0$ because $x^2+1>0$, $\forall x \in \mathbb{R}$.

	$x \in (-\infty, -1)$	$x \in (-1, 1)$	$x \in (1, \infty)$
$(x+1)(x-1)$	$(-)(-)$	$(+)(-)$	$(+)(+)$
	$\boxed{+}$	$-$	$\boxed{+}$

That is, the solution set for this case is $(1/2, \infty) \cap \{(-\infty, -1] \cup [1, \infty)\} = [1, \infty)$.

From(I), (II) and (III), The solution set is $\{1/2\} \cup \emptyset \cup [1, \infty) = \{\frac{1}{2}\} \cup [1, \infty)$. ■