

Outline

1. From Preferences to Utility (and viceversa)
2. Common utility functions
3. Utility maximization

1 From preferences to utility

- Nicholson, Ch. 3
- Economists like to use utility functions $u : X \rightarrow R$
- $u(x)$ is 'liking' of good x
- $u(a) > u(b)$ means: I prefer a to b .
- **Def.** Utility function u represents preferences \succeq if, for all x and y in X , $x \succeq y$ if and only if $u(x) \geq u(y)$.
- **Theorem.** If preference relation \succeq is rational and continuous, there exists a continuous utility function $u : X \rightarrow R$ that represents it.

- [Skip proof]

- Example:

$$(x_1, x_2) \succeq (y_1, y_2) \text{ iff } x_1 + x_2 \geq y_1 + y_2$$

- Draw:

- Utility function that represents it: $u(x) = x_1 + x_2$

- But... Utility function representing \succeq is not unique

- Take $3u(x)$ or $\exp(u(x))$

- $u(a) > u(b) \iff \exp(u(a)) > \exp(u(b))$

- If $u(x)$ represents preferences \succeq and f is a strictly increasing function, then $f(u(x))$ represents \succeq as well.

- If preferences are represented from a utility function, are they rational?
 - completeness
 - transitivity

- Indifference curves: $u(x_1, x_2) = \bar{u}$
- They are just implicit functions! $u(x_1, x_2) - \bar{u} = 0$

$$\frac{dx_2}{dx_1} = -\frac{U'_{x_1}}{U'_{x_2}} = MRS$$

- Indifference curves for:
 - monotonic preferences;
 - strictly monotonic preferences;
 - convex preferences

2 Common utility functions

- Nicholson, Ch. 3, pp. 102-105

1. Cobb-Douglas preferences: $u(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$

- $MRS = -\alpha x_1^{\alpha-1} x_2^{1-\alpha} / (1-\alpha) x_1^\alpha x_2^{-\alpha} = -\frac{\alpha}{1-\alpha} \frac{x_2}{x_1}$

2. Perfect substitutes: $u(x_1, x_2) = \alpha x_1 + \beta x_2$

- $MRS = -\alpha/\beta$

3. Perfect complements: $u(x_1, x_2) = \min(\alpha x_1, \beta x_2)$

- MRS discontinuous at $x_2 = \frac{\alpha}{\beta}x_1$

4. Constant Elasticity of Substitution: $u(x_1, x_2) = (\alpha x_1^\rho + \beta x_2^\rho)^{1/\rho}$

- $MRS = -\frac{\alpha}{\beta} \left(\frac{x_1}{x_2}\right)^{\rho-1}$
- if $\rho = 1$, then...
- if $\rho = 0$, then...
- if $\rho \rightarrow -\infty$, then...

3 Utility Maximization

- Nicholson, Ch. 4, pp. 119–128
- $X = R_+^2$ (2 goods)
- Consumers: choose bundle $x = (x_1, x_2)$ in X which yields highest utility.
- Constraint: income = M
- Price of good 1 = p_1 , price of good 2 = p_2
- Bundle x is feasible if $p_1x_1 + p_2x_2 \leq M$
- Consumer maximizes

$$\begin{aligned} & \max_{x_1, x_2} u(x_1, x_2) \\ & s.t. \quad p_1x_1 + p_2x_2 \leq M \\ & \quad x_1 \geq 0, \quad x_2 \geq 0 \end{aligned}$$

- Maximization subject to inequality. How do we solve that?
- Trick: u strictly increasing in at least one dimension. (\succeq strictly monotonic)
- Budget constraint always satisfied with equality
- Ignore temporarily $x_1 \geq 0, x_2 \geq 0$ and check afterwards that they are satisfied for x_1^* and x_2^* .

- Problem becomes

$$\begin{aligned} & \max_{x_1, x_2} u(x_1, x_2) \\ & s.t. \quad p_1 x_1 + p_2 x_2 - M = 0 \end{aligned}$$

- $L(x_1, x_2) = u(x_1, x_2) - \lambda(p_1 x_1 + p_2 x_2 - M)$

- F.o.c.s:

$$\begin{aligned} u'_{x_i} - \lambda p_i &= 0 \text{ for } i = 1, 2 \\ p_1 x_1 + p_2 x_2 - M &= 0 \end{aligned}$$

- Moving the two terms across and dividing, we get:

$$MRS = -\frac{u'_{x_1}}{u'_{x_2}} = -\frac{p_1}{p_2}$$

- Graphical interpretation.

- Second order conditions:

$$H = \begin{pmatrix} 0 & -p_1 & -p_2 \\ -p_1 & u''_{x_1,x_1} & u''_{x_1,x_2} \\ -p_2 & u''_{x_2,x_1} & u''_{x_2,x_2} \end{pmatrix}$$

$$\begin{aligned} |H| &= p_1 \left(-p_1 u''_{x_2,x_2} + p_2 u''_{x_2,x_1} \right) \\ &\quad - p_2 \left(-p_1 u''_{x_1,x_2} + p_2 u''_{x_1,x_1} \right) \\ &= -p_1^2 u''_{x_2,x_2} + 2p_1 p_2 u''_{x_1,x_2} - p_2^2 u''_{x_1,x_1} \end{aligned}$$

- Notice: $u''_{x_2,x_2} < 0$ and $u''_{x_1,x_1} < 0$ usually satisfied (but check it!).
- Condition $u''_{x_1,x_2} > 0$ is then sufficient

- Example with CES utility function.

$$\begin{aligned} \max_{x_1, x_2} & \left(\alpha x_1^\rho + \beta x_2^\rho \right)^{1/\rho} \\ \text{s.t.} & p_1 x_1 + p_2 x_2 - M = 0 \end{aligned}$$

- Lagrangean =

- F.o.c.:

- Solution:

$$x_1^* = \frac{M}{p_1 \left(1 + \left(\frac{\alpha}{\beta} \right)^{\frac{1}{\rho-1}} \left(\frac{p_2}{p_1} \right)^{\frac{\rho}{\rho-1}} \right)}$$

$$x_2^* = \frac{M}{p_2 \left(1 + \left(\frac{\beta}{\alpha} \right)^{\frac{1}{\rho-1}} \left(\frac{p_1}{p_2} \right)^{\frac{\rho}{\rho-1}} \right)}$$

- Special case 1: $\rho = 0$ (Cobb-Douglas)

$$x_1^* = \frac{\alpha}{\alpha + \beta} \frac{M}{p_1}$$

$$x_2^* = \frac{\beta}{\alpha + \beta} \frac{M}{p_2}$$

- Special case 1: $\rho \rightarrow 1^-$ (Perfect Substitutes)

$$x_1^* = \begin{cases} 0 & \text{if } p_1/p_2 \geq \alpha/\beta \\ M/p_1 & \text{if } p_1/p_2 < \alpha/\beta \end{cases}$$

$$x_2^* = \begin{cases} M/p_2 & \text{if } p_1/p_2 \geq \alpha/\beta \\ 0 & \text{if } p_1/p_2 < \alpha/\beta \end{cases}$$

- Special case 1: $\rho \rightarrow -\infty$ (Perfect Complements)

$$x_1^* = \frac{M}{p_1 + p_2} = x_2^*$$

- Parameter ρ indicates substitution pattern between goods:
 - $\rho > 0 \rightarrow$ Goods are (net) substitutes
 - $\rho < 0 \rightarrow$ Goods are (net) complements