Question 0: (Three-variable optimization)

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Suppose that there are three firms. Each firm produces one single product that is imperfectly substitutable to the other two products, i.e. differentiated products. Each of the firm *strategically* competes with each other, and faces the following market demand equations given by,

$$Q_1 = 80 - 2P_1 + P_{23}$$

$$Q_2 = 80 - 2P_2 + P_{13}$$

$$Q_3 = 80 - 2P_3 + P_{12}$$

where P_{23} is the average of the prices charged by firms 2 and 3, P_{13} is the average of the prices charged by firms 2, P_{12} is the average of the prices charged by firms 1 and 2 [e.g., $P_{12} = 0.5(P_1 + P_2)$]. Suppose that each of the three firms, as indexed by j, has the cost function given by $C^j(Q_j) = c_j Q_j$.

Consider the following problems.

a) Set up the profit function of each individual firm, and derive the profit-maximizing condition of each of the three firms when each of them *optimally* sets for the level of price that maximizes its own profit. (**Hint:** *Derive the best response function of each firm, i.e. optimal contingent plan or reaction function.*)

$$TR = P_1 \cdot Q_1$$
$$= P_1 \cdot (80 - 2P_1 + P_{23})$$
$$= P_1 \cdot (80 - 2P_1 + 0.5(P_2 + P_3))$$
$$= 80P_1 - 2P_1^2 + 0.5P_1P_2 + 0.5P_1P_3$$

$$TC = C^1(Q_1) = C_1 \cdot Q_1$$
$$= C_1 \cdot (80 - 2P_1 + 0.5(P_2 + P_3))$$
$$= 80C_1 - 2P_1C_1 + 0.5P_2C_1 + 0.5P_3C_1$$

$$\pi_1 = (80P_1 - 2P_1^2 + 0.5P_1P_2 + 0.5P_1P_3) - (80C_1 - 2P_1C_1 + 0.5P_2C_1 + 0.5P_3C_1)$$

$$\frac{d\pi_1}{dP_1} = 80 - 4P_1 + 0.5P_2 + 0.5P_3 + 2C_1$$

$$TR = P_2 \cdot Q_2$$
$$= P_2(80 - 2P_2 + P_{13})$$
$$= P_2(80 - 2P_2 + 0.5(P_1 + P_3))$$
$$= 80P_2 - 2P_2^2 + 0.5P_1P_2 + 0.5P_2P_3$$

$$TC = C^2(Q_2) = C_2 \cdot Q_2$$
$$= C_2(80 - 2P_2 + 0.5(P_1 + P_3))$$
$$= 80C_2 - 2P_2C_2 + 0.5P_1C_2 + 0.5P_3C_2$$

$$\pi_2 = (80P_2 - 2P_2^2 + 0.5P_1P_2 + 0.5P_2P_3) - (80C_2 - 2P_2C_2 + 0.5P_1C_2 + 0.5P_3C_2)$$

$$\frac{d\pi_2}{dP_2} = 80 - 4P_2 + 0.5P_1 + 0.5P_3 + 2C_2$$

$$\begin{aligned} TR &= P_3 \cdot Q_3 \\ &= P_3(80 - 2P_3 + P_{12}) \\ &= P_3(80 - 2P_3 + 0.5(P_1 + P_2)) \\ &= 80P_3 - 2P_3^2 + 0.5P_1P_3 + 0.5P_2P_3 \end{aligned}$$

$$\begin{aligned} TC = C^3(Q_3) &= C_3Q_3 \\ &= C_3(80 - 2P_3 + 0.5(P_1 + P_2)) \\ &= 80C_3 - 2P_3C_3 + 0.5P_1C_3 + 0.5P_2C_3 \end{aligned}$$

$$\pi_3 = (80P_3 - 2P_3^2 + 0.5P_1P_3 + 0.5P_2P_3) - (80C_3 - 2P_3C_3 + 0.5P_1C_3 + 0.5P_2C_3)$$

$$\frac{d\pi_3}{dP_3} = 80 - 4P_3 + 0.5P_1 + 0.5P_2 + 2C_3$$

$$\frac{d\pi_1}{dP_1} = 80 - 4P_1 + 0.5P_2 + 0.5P_3 + 2C_1 = 0$$

$$\frac{d\pi_2}{dP_2} = 80 - 4P_2 + 0.5P_1 + 0.5P_3 + 2C_2 = 0$$

$$\frac{d\pi_3}{dP_3} = 80 - 4P_3 + 0.5P_1 + 0.5P_2 + 2C_3 = 0$$

$$P_1 = \frac{80 + 0.5P_2 + 0.5P_3 + 2C_1}{4} \rightarrow BR_1(P_2, P_3)$$

$$P_2 = \frac{80 + 0.5P_1 + 0.5P_3 + 2C_2}{4} \rightarrow BR_2(P_1, P_3)$$

$$P_3 = \frac{80 + 0.5P_1 + 0.5P_2 + 2C_3}{4} \rightarrow BR_3(P_1, P_2)$$

$$\underbrace{\begin{bmatrix} 4 & -0.5 & -0.5 \\ -0.5 & 4 & -0.5 \\ -0.5 & -0.5 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 80+2C_1 \\ 80+2C_2 \\ 80+2C_3 \end{bmatrix}}_{D}$$

$$X = \begin{bmatrix} 26.44 + 0.5142C_1 + 0.0734C_2 + 0.0734C_3 \\ 26.44 + 0.0734C_1 + 0.5142C_2 + 0.0734C_3 \\ 26.44 + 0.0734C_1 + 0.0734C_2 + 0.5142C_3 \end{bmatrix}$$

$$P_1^* = 26.44 + 0.5142C_1 + 0.0734C_2 + 0.0734C_3$$
$$P_2^* = 26.44 + 0.0734C_1 + 0.5142C_2 + 0.0734C_3$$
$$P_3^* = 26.44 + 0.0734C_1 + 0.0734C_2 + 0.5142C_3$$

b) Suppose that Based on (a), what would be the *equilibrium* level of price that each of the firms choose. Calculate the level of profit that each firm yields. (Hint: *Solve for the Bertrand equilibrium with differentiated product*)

$$P_1^* = 26.44 + 0.5142C_1 + 0.0734C_2 + 0.0734C_3$$
$$P_2^* = 26.44 + 0.0734C_1 + 0.5142C_2 + 0.0734C_3$$
$$P_3^* = 26.44 + 0.0734C_1 + 0.0734C_2 + 0.5142C_3$$

$$\pi_1^* = P_1^* \cdot Q_1^*$$
$$= (26.44 + 0.5142C_1 + 0.0734C_2 + 0.0734C_3)(80 - 2P_1 + 0.5(P_2 + P_3))$$

$$\pi_2^* = P_2^* \cdot Q_2^*$$
$$= (26.44 + 0.0734C_1 + 0.5142C_2 + 0.0734C_3)(80 - 2P_2 + 0.5(P_1 + P_3))$$

$$\pi_3^* = P_3^* \cdot Q_3^*$$
$$= (26.44 + 0.0734C_1 + 0.0734C_2 + 0.5142C_3)(80 - 2P_3 + 0.5(P_1 + P_2))$$

c) How does the optimal price vary with respect to the size of marginal cost?

$$\frac{\partial P_1^*}{\partial C_1} = 0.5142$$

$$\frac{\partial P_2^*}{\partial C_2} = 0.5142$$

$$\frac{\partial P_3^*}{\partial C_3} = 0.5142$$

$\therefore$ when the MC increases by 1 unit, optimal price increases by 0.5142 units.

Now suppose that all the three firms agree to operate under a cartel (collusive) agreement. That is, they consider a joint pricing scheme that maximizes the joint profit function of the three firms combined. Consider the following problems.

d) Construct the joint profit function.

$$\Pi = P_1 \cdot Q_1 + P_2 \cdot Q_2 + P_3 \cdot Q_3 - C_1 Q_1 - C_2 Q_2 - C_3 Q_3$$

$$= (80P_1 - 2P_1^2 + 0.5P_1P_2 + 0.5P_1P_3) + (80P_2 - 2P_2^2 + 0.5P_1P_2 + 0.5P_2P_3)$$

$$+ (80P_3 - 2P_3^2 + 0.5P_1P_3 + 0.5P_2P_3) - (80C_1 - 2P_1C_1 + 0.5P_2C_1 + 0.5P_3C_1)$$

$$-(80C_2 - 2P_2C_2 + 0.5P_1C_2 + 0.5P_3C_2) - (80C_3 - 2P_3C_3 + 0.5P_1C_3 + 0.5P_2C_3)$$

e) Given that $c_j = c,\ \forall j$. Calculate the level of optimal price setting that each of the firm will set under the joint profit function problem. Confirm your result with the second-order derivative test.

$$\frac{\partial \pi}{\partial P_1} = 80 - 4P_1 + 0.5P_2 + 0.5P_3 + 0.5P_2 + 0.5P_3 + 2C_1 - 0.5C_2 - 0.5C_3$$
$$= 80 - 4P_1 + P_2 + P_3 + 2C_1 - 0.5C_2 - 0.5C_3$$

$$\frac{\partial \pi}{\partial P_2} = 0.5P_1 + 80 - 4P_2 + 0.5P_1 + 0.5P_3 + 0.5P_3 - 0.5C_1 + 2C_2 - 0.5C_3$$
$$= 80 - 4P_2 + P_1 + P_3 - 0.5C_1 + 2C_2 - 0.5C_3$$

$$\frac{\partial \pi}{\partial P_3} = 0.5P_1 + 0.5P_2 + 80 - 4P_3 + 0.5P_1 + 0.5P_2 - 0.5C_1 - 0.5C_2 + 2C_3$$
$$= 80 - 4P_3 + P_1 + P_2 - 0.5C_1 - 0.5C_2 + 2C_3$$

$$80 - 4P_1 + P_2 + P_3 + 2C_1 - 0.5C_2 - 0.5C_3 = 0 \quad \text{——— (1)}$$
$$80 - 4P_2 + P_1 + P_3 - 0.5C_1 + 2C_2 - 0.5C_3 = 0 \quad \text{——— (2)}$$
$$80 - 4P_3 + P_1 + P_2 - 0.5C_1 - 0.5C_2 + 2C_3 = 0 \quad \text{——— (3)}$$

(1) = (2) ; $4P_1 - P_2 - P_3 = -P_1 + 4P_2 - P_3$

$$P_1 = P_2$$

Plug $P_1 = P_2$ into (1) & (3)

$$4P_1 - P_1 - P_3 = -P_1 - P_1 + 4P_3$$
$$P_1 = P_3$$

Plug $P_1 = P_2 = P_3$ into (1)

$$4P_1 - P_1 - P_1 = 80 - C$$
$$P_1^* = P_2^* = P_3^* = \frac{80 - C}{3}$$

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$$H = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} -4 & 1 & 1 \\ 1 & -4 & 1 \\ 1 & 1 & -4 \end{bmatrix}$$

$|H_1| = -4 < 0$

$$|H_2| = \begin{vmatrix} -4 & 1 \\ 1 & -4 \end{vmatrix} = (-4)(-4)-(1)(1) = 15 > 0$$

$$|H_3| = \begin{vmatrix} -4 & 1 & 1 \\ 1 & -4 & 1 \\ 1 & 1 & -4 \end{vmatrix} \begin{matrix} -4 & 1 \\ 1 & -4 \\ 1 & 1 \end{matrix} = -50 < 0$$

$\therefore$ H is negative definite $\rightarrow d^2\pi < 0 \rightarrow$ globally concave

P_1^*, P_2^*, P_3^* are the global solutions.

f) Is the Cartel agreement self-sustaining? Why?

Cartel agreement is not self-sustaining even though all 3 producers of the firms are different. The Qd of each good is interdependent on one another. If one of the firm were to set price a little bit higher than it is in the agreement, the Qd for other producers will drop. Therefore, the cheated firm will gain relatively higher profit compared to the rest.

g) Revisit question (e) and assume that the marginal costs are varied with respect to firms. What would be the implication of the collusive equilibrium?

The implication of collusive eqm. would be that if MC are varied wrt. firms, they should set the price of each product based on MC of all 3 goods. That is, if the price of product 1 has higher MC, holding all other things constant, the firm should sell it at relatively. At the end, they should satisfy the condition where $MR(Q_1^*) = MR(Q_2^*) = MR(Q_3^*) = MC(Q_{TOTAL})$

Question 1:

Given the production function

$$Q = f(K, L) = K^{0.5}L^{0.5}$$

Suppose that the per unit input prices for K and L are \$20 and \$5, respectively.

a. If the producer needs to maintain the output level at $\underline{Q} = 20$ units, what are the values K^* *and* L^* *that minimizes the total cost*, and what is the corresponding minimum cost? Use the bordered Hessian to verify that the second-order sufficient condition is met.

- Total cost $= wL + rk$ $\quad$ s.t. $\quad 20 = L^{0.5}k^{0.5}$

$= 5L + 20k$

$$\mathcal{L}(k, L, \lambda; r, w) = 20k + 5L + \lambda(L^{0.5}k^{0.5} - 20)$$

$$\frac{\partial \mathcal{L}}{\partial k} = 20 + \frac{1}{2}L^{0.5}k^{-\frac{1}{2}}\lambda = 0 \; ; \quad L^{\frac{1}{2}}k^{-\frac{1}{2}} = \frac{-20 \times 2}{\lambda} = \frac{-40}{\lambda}$$

$$L^{\frac{1}{2}} = \frac{-40}{\lambda k^{-\frac{1}{2}}} =$$

$$L = \frac{1600}{\lambda^2 k^{-1}}$$

$$L = \frac{1600k}{\lambda^2} \quad \text{— (1)}$$

$$\frac{\partial \mathcal{L}}{\partial L} = 5 + \frac{1}{2}L^{-\frac{1}{2}}k^{0.5}\lambda = 0 \; ; \quad L^{-\frac{1}{2}}k^{\frac{1}{2}} = \frac{-10}{\lambda}$$

$$\left(k^{\frac{1}{2}}\right)^2 = \left(\frac{-10}{\lambda L^{-\frac{1}{2}}}\right)^2$$

$$k = \frac{100L}{\lambda^2} \quad \text{— (2)}$$

$$\frac{k(\lambda^2)}{100} = L \quad \text{— (4)}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = L^{\frac{1}{2}}k^{\frac{1}{2}} - 20 = 0 \; ; \quad L^{\frac{1}{2}}k^{\frac{1}{2}} = 20$$

$$\rightarrow L^{\frac{1}{2}} = \frac{20}{k^{\frac{1}{2}}}$$

$$\left(L^{\frac{1}{2}}\right)^2 = \frac{400}{k}$$

$$L = \frac{400}{k} \quad \text{— (3)}$$

(3) into (2);

$$k = \frac{100\left(\frac{400}{k}\right)}{\lambda^2}$$

$$\lambda^2 = \frac{\frac{40000}{k}}{k}$$

$$\lambda^* = \frac{200}{k}$$

(1) = (4);

$$\frac{1600k}{\lambda^2} = \frac{k(\lambda^2)}{100}$$

$$160000k = k(\lambda^2)(\lambda^2)$$

$$160{,}000k = \cancel{k}\left(\frac{40{,}000}{\cancel{k^2}\,k}\right)\left(\frac{40{,}000}{k^2}\right)$$

$$160{,}000k = \frac{1{,}600{,}000}{k^3}$$

$$160{,}000k^4 = 1{,}600{,}000$$

$$k^4 = 10{,}000$$

$$k^* = 10$$

k^* into (3);

$$L = \frac{400}{(10)}$$

$$L^* = 40$$

SOC : $\bar{H} = \begin{bmatrix} L_{\lambda\lambda} & L_{\lambda K} & L_{\lambda L} \\ L_{K\lambda} & L_{KK} & L_{KL} \\ L_{L\lambda} & L_{LK} & L_{LL} \end{bmatrix}$

$$L_{\lambda\lambda} = \frac{\partial\, L^{\frac{1}{2}}K^{\frac{1}{2}} - 20}{\partial \lambda}$$

$$= 0$$

$$L_{K\lambda} = \frac{\partial\, 20 + \frac{1}{2}L^{0.5}K^{-\frac{1}{2}}\lambda}{\partial \lambda}$$

$$= \frac{1}{2}L^{\frac{1}{2}}K^{-\frac{1}{2}}$$

$$L_{L\lambda} = \frac{\partial\, 5 + \frac{1}{2}L^{-\frac{1}{2}}K^{\frac{1}{2}}\lambda}{\partial \lambda}$$

$$= \frac{1}{2}L^{-\frac{1}{2}}K^{\frac{1}{2}}$$

$$L_{KK} = \frac{\partial\, 20 + \frac{1}{2}L^{0.5}K^{-\frac{1}{2}}\lambda}{\partial K}$$

$$= -\frac{1}{4}L^{\frac{1}{2}}K^{-\frac{3}{2}}\lambda$$

$$L_{LK} = \frac{\partial\, 5 + \frac{1}{2}L^{-\frac{1}{2}}K^{\frac{1}{2}}\lambda}{\partial K}$$

$$= \frac{1}{4}L^{-\frac{1}{2}}K^{-\frac{1}{2}}\lambda$$

$$L_{LL} = \frac{\partial\, 5 + \frac{1}{2}L^{-\frac{1}{2}}K^{\frac{1}{2}}\lambda}{\partial L}$$

$$= -\frac{1}{4}L^{-\frac{3}{2}}K^{\frac{1}{2}}\lambda$$

$$\bar{H} = \begin{bmatrix} 0 & \frac{1}{2}L^{\frac{1}{2}}K^{-\frac{1}{2}} & \frac{1}{2}L^{-\frac{1}{2}}K^{\frac{1}{2}} \\ \frac{1}{2}L^{\frac{1}{2}}K^{-\frac{1}{2}} & -\frac{1}{4}L^{\frac{1}{2}}K^{-\frac{3}{2}}\lambda & \frac{1}{4}L^{-\frac{1}{2}}K^{-\frac{1}{2}}\lambda \\ \frac{1}{2}L^{-\frac{1}{2}}K^{\frac{1}{2}} & \frac{1}{4}L^{-\frac{1}{2}}K^{-\frac{1}{2}}\lambda & -\frac{1}{4}L^{-\frac{3}{2}}K^{\frac{1}{2}}\lambda \end{bmatrix}$$

$N = 2$

$$|\bar{H}_3| = -\frac{1}{4}L^{-\frac{1}{2}}K^{-\frac{1}{2}}\lambda$$

$|\bar{H}|$ is negative. Total cost is convex along the constaint

b. Suppose now that the output level is not fixed, but the producer has a budget constraint at \$400. Assume that this producer spends his entire budget on the production. Determine the values K^* *and* L^* *that maximizes the total output.* Also, use the bordered Hessian to verify that the second-order sufficient condition is met.

$$\max_{K,L} Q = L^{0.5}K^{0.5} \qquad \text{s.t.} \quad 400 = wL + rK$$
$$= 5L + 20K$$

$$\mathcal{L}(K, L, \lambda; r, w) = L^{0.5}K^{0.5} + \lambda(400 - wL - rK)$$

$$\frac{\partial \mathcal{L}}{\partial K} = \frac{1}{2}L^{0.5}K^{-\frac{1}{2}} - \lambda r = 0;$$
$$\frac{1}{2}L^{0.5}K^{-\frac{1}{2}} = \lambda r$$

$$\frac{r}{w} = \frac{MP_K}{MP_L} = \frac{\frac{\frac{1}{2}L^{0.5}K^{-\frac{1}{2}}}{\lambda}}{\frac{\frac{1}{2}L^{-\frac{1}{2}}K^{0.5}}{\lambda}} = \frac{L^{0.5}K^{-\frac{1}{2}}}{L^{-\frac{1}{2}}K^{0.5}} = \frac{L}{K}$$

$$\frac{\partial \mathcal{L}}{\partial L} = \frac{1}{2}L^{-\frac{1}{2}}K^{0.5} - \lambda w = 0;$$
$$\frac{1}{2}L^{-\frac{1}{2}}K^{0.5} = \lambda w$$

$$\frac{r}{w} = \frac{L}{K}$$
$$L = K\left(\frac{r}{w}\right)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 400 - wL - rK = 0;$$
$$0 = 400 - w\left(K\left(\frac{r}{w}\right)\right) - rK$$
$$0 = 400 - wrK - rK$$
$$K(wr - r) = 400$$
$$K^* = \frac{400}{wr - r}$$

$$L^* = \frac{400}{wr - r}\left(\frac{r}{w}\right)$$
$$= \frac{400\,\cancel{(r)}}{\cancel{(r)}(w-1)}$$
$$L^* = \frac{400}{(w-1)}$$

$$\text{SOC}: \bar{H} = \begin{bmatrix} L_{\lambda\lambda} & L_{\lambda K} & L_{\lambda L} \\ L_{K\lambda} & L_{KK} & L_{KL} \\ L_{L\lambda} & L_{LK} & L_{LL} \end{bmatrix}$$

$$L_{\lambda\lambda} = \frac{\partial\, 400 - wL - rK}{\partial \lambda} = 0$$

$$L_{K\lambda} = \frac{\partial\, \frac{1}{2}L^{0.5}K^{-\frac{1}{2}} - \lambda r}{\partial \lambda} = -r$$

$$L_{L\lambda} = \frac{\partial\, \frac{1}{2}L^{-\frac{1}{2}}K^{0.5} - \lambda w}{\partial \lambda} = -w$$

$$L_{KK} = \frac{\partial\, \frac{1}{2}L^{0.5}K^{-\frac{1}{2}} - \lambda r}{\partial K} = -\frac{1}{4}L^{\frac{1}{2}}K^{-\frac{3}{2}}$$

$$L_{LK} = \frac{\partial\, \frac{1}{2}L^{-\frac{1}{2}}K^{0.5} - \lambda w}{\partial K} = \frac{1}{4}L^{-\frac{1}{2}}K^{-\frac{1}{2}}$$

$$L_{LL} = \frac{\partial\, \frac{1}{2}L^{-\frac{1}{2}}K^{0.5} - \lambda w}{\partial L} = -\frac{1}{4}LK^{\frac{1}{2}}$$

$$\bar{H} = \begin{bmatrix} 0 & -r & -w \\ -r & -\frac{1}{4}L^{\frac{1}{2}}K^{-\frac{3}{2}} & \frac{1}{4}L^{-\frac{1}{2}}K^{-\frac{1}{2}} \\ -w & \frac{1}{4}L^{-\frac{1}{2}}K^{-\frac{1}{2}} & -\frac{1}{4}LK^{\frac{1}{2}} \end{bmatrix}$$

$N = 2$

$$|\bar{H}| = 0 - (r^2)$$
$$= -r^2$$

$|\bar{H}|$ is negative. Total output is convex along the constaint.

Question 3:

Consider a two-period consumption model. The consumer's utility is given by:

$$U(C_1, C_2) = \frac{C_1^{1-\theta}}{1-\theta} + \beta \frac{C_2^{1-\theta}}{1-\theta},$$

where C_1 and C_2 are consumption levels in period 1 and period 2, respectively; θ measures the degree of relative risk aversion where $0 < \theta < 1$; and β is the discount factor where $0 < \beta < 1$.

The consumer's budget constraint is given by:

$$C_1 + \frac{C_2}{1+r} = I_1,$$

where r is the interest rate ($0 < r < 1$), and I_1 is the income in period 1.

Answer the following questions.

a) Write down the utility maximization problem subject to the above budget constraint. Use Lagrange method to derive the optimal values of C_1^* and C_2^* in terms of the parameters θ, β, r, and I_1.

b) Based on your answer in (3.1), derive the impact of an increase in the interest rate on the optimal consumption in period 1 ($\frac{\partial C_1^*}{\partial r}$). Interpret its economic meaning.

c) Verify your answer in (a) by using the second-order sufficient condition.

(a)

$$\max_{C_1, C_2} \frac{C_1^{1-\theta}}{1-\theta} + \beta \frac{C_2^{1-\theta}}{1-\theta} \quad \text{s.t. } C_1 + \frac{C_2}{1+r} = I_1$$

$$0 = I_1 - C_1 - \frac{C_2}{1+r}$$

$$\mathcal{L}(C_1, C_2, \lambda; \theta, \beta, r, I_1)$$

$$= \frac{C_1^{1-\theta}}{1-\theta} + \beta \frac{C_2^{1-\theta}}{1-\theta} + \lambda\left(I_1 - C_1 - \frac{C_2}{1+r}\right)$$

$$C_2^{-\theta} = \frac{\lambda}{(1+r)\beta}$$

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$$[C_1]: \frac{\partial \mathcal{L}}{\partial C_1} = C_1^{-\theta} - \lambda = 0 \Rightarrow C_1^{-\theta} = \lambda \quad -(1)$$

$$[C_2]: \frac{\partial \mathcal{L}}{\partial C_2} = \beta C_2^{-\theta} - \frac{\lambda}{1+r} = 0 \Rightarrow \beta C_2^{-\theta} = \frac{\lambda}{1+r} \quad -(2)$$

$$[\lambda]: \frac{\partial \mathcal{L}}{\partial \lambda} = I_1 - C_1 - \frac{C_2}{1+r} = 0 \Rightarrow C_1 + \frac{C_2}{1+r} = I_1 \quad -(3)$$

$$\frac{(1)}{(2)} \quad \frac{C_1^{-\theta}}{\beta C_2^{-\theta}} = \frac{\lambda}{\frac{\lambda}{1+r}}$$

$$\frac{C_1^{-\theta}}{C_2^{-\theta}} = (1+r)\beta$$

$$C_1 = \left[(1+r)\beta C_2^{-\theta}\right]^{\theta}$$

$$C_1 = \left[(1+r)\beta\right]^{\theta} \cdot C_2$$

$$C_2 = \frac{C_1}{\left[(1+r)\beta\right]^{\theta}}$$

$$\therefore C_2 = \frac{\frac{I_1 (1+r)^{\theta+1}\beta}{(1+r)^{\theta+1}\beta^{\theta}+1}}{\left[(1+r)\beta\right]^{\theta}}$$

$$C_2^{*} = \frac{I_1 (1+r)^{\theta}\beta}{(1+r)^{\theta+1}\beta^{\theta}+1}$$

Plug into (3)

$$C_1 + \frac{\frac{C_1}{(1+r)^{\theta}\beta^{\theta}}}{(1+r)} = I_1$$

$$C_1 + \frac{C_1}{(1+r)^{\theta+1}\beta^{\theta}} = I_1$$

$$C_1\left(1 + \frac{1}{(1+r)^{\theta+1}\beta^{\theta}}\right) = I_1$$

$$C_1^{*} = \frac{I_1}{1 + \frac{1}{(1+r)^{\theta+1}\beta^{\theta}}}$$

$$= \frac{I_1}{\frac{(1+r)^{\theta+1}\beta^{\theta}+1}{(1+r)^{\theta+1}\beta}}$$

$$C_1^{*} = \frac{I_1 (1+r)^{\theta+1}\beta}{(1+r)^{\theta+1}\beta^{\theta}+1}$$

b) $$\frac{\partial C_1^*}{\partial r} = \frac{\partial}{\partial r} \frac{I_1 (1+r)^{\theta+1} \beta}{(1+r)^{\theta+1}\beta^{\theta}+1}$$

$$= \frac{\left[(1+r)^{\theta+1}\beta^{\theta}+1\right]\left[\beta I_1 (\theta+1)(1+r)^{\theta}(2)\right] - \left[I_1(1+r)^{\theta+1}\beta\right]\left[(\theta+1)(1+r)^{\theta}(2)\beta^{\theta}\right]}{\left[(1+r)^{\theta+1}\beta^{\theta}+1\right]^2}$$

$$= \frac{2\beta I_1(\theta+1)(1+r)^{\theta}\left[\overbrace{\cancel{(1+r)^{\theta+1}\beta^{\theta}}+1-\cancel{(1+r)^{\theta+1}\beta^{\theta}}}^{=1}\right]}{\left[(1+r)^{\theta+1}\beta^{\theta}+1\right]^2}$$

$$\frac{\partial C_1^*}{\partial r} = \frac{2\beta I_1(\theta+1)(1+r)^{\theta}}{\left[(1+r)^{\theta+1}\beta^{\theta}+1\right]^2}$$

when r increases by 1 unit

the optimal consumption in period 1 (C_1) increases

by $\dfrac{2\beta I_1(\theta+1)(1+r)^{\theta}}{\left[(1+r)^{\theta+1}\beta^{\theta}+1\right]^2}$ units.

c) From $\xrightarrow{\text{F.O.C}}$

$$[C_1]: \frac{\partial L}{\partial C_1} = C_1^{-\theta} - \lambda = 0$$

$$[C_2]: \frac{\partial L}{\partial C_2} = \beta C_2^{-\theta} - \frac{\lambda}{1+r} = 0$$

$$[\lambda]: \frac{\partial L}{\partial \lambda} = I_1 - C_1 - \frac{C_2}{1+r} = 0$$

<u>S.O.C</u>

$$\bar{H} = \begin{bmatrix} L_{\lambda\lambda} & L_{\lambda C_1} & L_{\lambda C_2} \\ L_{C_1\lambda} & L_{C_1C_1} & L_{C_1C_2} \\ L_{C_2\lambda} & L_{C_2C_1} & L_{C_2C_2} \end{bmatrix} = \begin{bmatrix} 0 & -1 & -\frac{1}{1+r} \\ -1 & -\theta C_1^{-\theta-1} & 0 \\ -\frac{1}{1+r} & 0 & -\theta\beta C_2^{-\theta-1} \end{bmatrix}$$

$$|\bar{H}| = \left| \begin{array}{ccc|cc} 0 & -1 & -\frac{1}{1+r} & 0 & -1 \\ -1 & -\theta C_1^{-\theta-1} & 0 & -1 & -\theta C_1^{-\theta-1} \\ -\frac{1}{1+r} & 0 & -\theta\beta C_2^{-\theta-1} & -\frac{1}{1+r} & 0 \end{array} \right.$$

$$= 0 - \left[\left(-\frac{1}{1+r}\right)\left(-\theta C_1^{-\theta-1}\right)\left(-\frac{1}{1+r}\right) + 1\left(-\theta\beta C_2^{-\theta-1}\right) \right]$$

$$= -\left[\left(\frac{1}{1+r}\right)^2 \left(-\theta C_1^{-\theta-1}\right) - \theta\beta C_2^{-\theta-1} \right]$$

$$= \left(\frac{1}{1+r}\right)^2 \left(\theta C_1^{-\theta-1}\right) + \theta\beta C_2^{-\theta-1} > 0$$

$U(C_1, C_2)$ is concave along the constraint set

$\therefore$ Solution from ⓐ are maximizers.

Question 3:

Pakorn's utility function depends on the consumption of two commodities, x and y, and it is given by

$$U(x, y) = 2xy$$

Suppose that his income is \$72, and the prices per unit of x and y are \$4 and \$6, respectively. Assume that Pakorn spends all of his income, and the values of x and y are both non-zero.

d. Use the Lagrange method to determine the values of x^* and y^* that maximize Pakorn's utility given an income constraint. Verify that the second-order sufficient conditions are satisfied.
e. Determine the maximum utility level and the Lagrange multiplier. Interpret the economic interpretation of the Lagrange multiplier.
f. Suppose that the income is now \$73. Approximate the new maximum utility level.

d. Determine the value of x^* and y^* that maximize Pakorn's utility given an income constraint

$$\max_{x,y} \quad U(x,y) = 2xy$$
$$\text{s.t.} \quad 4x + 6y = 72$$

Lagrange method

$$L = 2xy + \lambda(72 - 4x - 6y)$$

F.O.C

$$[x]: \quad \frac{\partial L}{\partial x} = 2y + \lambda(-4) = 0$$
$$2y = 4\lambda \Rightarrow y = 2\lambda \quad (1)$$

$$[y]: \quad \frac{\partial L}{\partial y} = 2x + \lambda(-6) = 0$$
$$2x = 6\lambda \Rightarrow x = 3\lambda \quad (2)$$

$$[\lambda]: \quad \frac{\partial L}{\partial \lambda} = 72 - 4x - 6y = 0$$
$$4x + 6y = 72 \quad (3)$$

Substitute (1) and (2) into (3) :

$$4x + 6y = 72$$

$$4(3\lambda) + 6(2\lambda) = 72$$

$$12\lambda + 12\lambda = 72$$

$$24\lambda = 72$$

$$\lambda^* = \frac{72}{24} = 3 \text{ units}$$

$$y = 2\lambda$$

$$y^* = 2 \times 3 = 6 \text{ units}$$

$$x = 3\lambda$$

$$x^* = 3 \times 3 = 9 \text{ units}$$

$$(x^*, y^*, \lambda^*) = (9, 6, 3)$$

<u>S.O.C</u>

$$\bar{H} = \begin{bmatrix} L_{\lambda\lambda} & L_{\lambda x} & L_{\lambda y} \\ L_{x\lambda} & L_{xx} & L_{xy} \\ L_{y\lambda} & L_{yx} & L_{yy} \end{bmatrix} = \begin{bmatrix} 0 & -4 & -6 \\ -4 & 0 & 2 \\ -6 & 2 & 0 \end{bmatrix}$$

$$|\bar{H}| = \left|\begin{array}{ccc|cc} 0 & -4 & -6 & 0 & -4 \\ -4 & 0 & 2 & -4 & 0 \\ -6 & 2 & 0 & -6 & 2 \end{array}\right.$$

$$|\bar{H}| = [0 + (-4)(2)(-6) + (-6)(-4)(2)] - [0 + 0 + 0] = 96$$

$\hookrightarrow |\bar{H}| > 0 \rightarrow f$ is concave along the constraint set

$\rightarrow$ Solution from F.O.C represents maximizer.

e. Determine the maximum utility level and the Lagrange multiplier

$U(x, y) = 2xy$

$U^*(9, 6) = 2 \times 9 \times 6 = 108$ utils

$$\frac{dU^*}{dc} = \lambda^* = 3 > 0$$

When c increases by 1 unit, U^* increases by 3 utils.

or when his income increases by 1\$, his maximum utility level increases by 3 utils.

f. Suppose that income is now \$ 73

Since his old income is \$ 72, his income has increased by \$ 1.

$$\Rightarrow \Delta U^* = \Delta C \times \lambda^*$$

$$\Delta U^* = 1 \times 3 = 3 \text{ utils}$$

$U^*_{new} = 108 + 3 = 111$ utils