The Logic of Compound Statements: I

Outline:

- Introduction: Compound Statements
- Truth Table
- Evaluating the Truth of Compound Statements
- Logical Equivalence: De Morgan's Law; Tautology & Contradiction
- Conditional & Biconditional Statements

Mathematical logic is a tool for dealing with formal reasoning. Most of the definitions of formal logic have been developed so that they agree with the natural or intuitive logic used by people who have been educated to think clearly and use language carefully. The differences that exist between formal and intuitive logic are necessary to avoid ambiguity and obtain consistency.

1 Statements

In any mathematical theory, new terms are defined by using those that have been previously defined. However, this process has to start somewhere. A few initial terms necessarily remain undefined. In logic, the words **sentence**, **true**, and **false** are the initial undefined terms.

Definition 1.1 (Statements/ Propositions)**.** :
A **statement** (or **proposition**) is a sentence that is *true* or *false* but not both.

Example 1.1. Which of the following are statements ? Give the truth value for each of the statements.

(1) $1 + 1 = 0$

(2) $1 > 2$

(3) $1 + 1$

(4) $x - y$

(5) What time is it?

(6) Stop!

Solutions:

- The statements are

- The sentences that are not statements are

1.1 Compound Statements

- When a new statement (proposition) is constructed from two or more statements, we call this new statement as a *" compound statement"*.
- A compound statements are built by connecting more than two statements by using the following *operations of logical connectives* :

symbol	$\sim$	$\wedge$	$\vee$
meaning	"not"	"and"	"or"

Let p and q be statements. The following are compound statements constructed from the above operations:

- Negation of p: denote by $\boxed{\sim p}$ (read "not p")
- Conjunction of p and q: denoted by $\boxed{p \wedge q}$ (read "p and q")
- Disjunction of p and q: denoted by $\boxed{p \vee q}$ (read "p or q")

The order of operations

1. $\sim$
2. $\wedge$ and $\vee$ (coequal)

Note: The order of operations can be overridden through the use of parentheses. E.g.

- $\sim p \wedge q = (\sim p) \wedge q$.
- $\sim (p \wedge q)$ is the negation of p and q.
- $p \wedge q \vee r$ is considered ambiguous (must be either $(p \wedge q) \vee r$ or $p \wedge (q \vee r)$).

Example 1.2. suppose x is a particular real number. Let
p be the statement for $0 < x$,
q be the statement for $x < 3$,
r be the statement for $x = 3$.
Write the following inequalities symbolically by using p, q, and r.

(i) $x \leq 3$

(ii) $0 < x < 3$

(iii) $0 < x \leq 3$

Solution:

Definition 1.2 (Statement Variable,statement form, Truth table)**.** :

- A **statement variable** is a variable of statement whose value can be true (T) or false (F)
- A **statement form** (or **propositional form**) is an expression made up of statement variables (e.g. p, q, and r) and logical connectives (e.g. $\sim, \wedge, \vee$) that becomes a statement when actual statements are substituted for the component statement variables.
- The **truth table** for a given statement form displays the truth values that correspond to *all possible combinations* of truth values for its component statement variables.

  Note: If a given statement form consists of n statement variables, then there are 2^n rows in the truth table.

1.1.1 Truth Value

Let p and q be statement variables.

- Negation of p:"not" $\boxed{\sim p}$
  Truth Table:

p	$\sim p$
T	F
F	T

- Conjunction of p and q: "and" $\boxed{p \wedge q}$
  Truth table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

- Disjunction of p and q: "or" $\boxed{p \vee q}$
  Truth table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

1.1.2 Evaluating the Truth of Compound Statements

Example 1.3. Construct the truth table for the statement form :

$$(p \vee q) \wedge \sim (p \wedge q).$$

Note that the above is also called *exclusive or* and sometime denoted by $p \oplus q$ or p XOR q.
Solution:

- We have to consider the total of 4 possible combinations of truth values of p and q (since each of p and q can be either true of false).
- We will consider the truth values of:
p, q, $p \vee q$, $p \wedge q$, $\sim (p \wedge q)$ and $(p \vee q) \wedge \sim (p \wedge q)$.

Truth table:

p	q	$p \vee q$	$p \wedge q$	$\sim (p \wedge q)$	$(p \vee q) \wedge \sim (p \wedge q)$
T	T				
T	F				
F	T				
F	F				

■

Example 1.4. Construct the truth table for the statement form:

$$(p \wedge q) \vee \sim r.$$

Solution:

- Since there are 3 statement variables (p, q, r) and each of them can have 2 truth vales('T' or 'F'), then there are **8** possible combinations of truth values.
- In the truth table, we will consider the truth values of
p, q, r, $p \wedge q$, $\sim r$, and $(p \wedge q) \vee \sim r$.

Truth table:

p	q	r	$p \wedge q$	$\sim r$	$(p \wedge q) \vee \sim r.$
T	T	T			
T	T	F			
T	F	T			
T	F	F			
F	T	T			
F	T	F			
F	F	T			
F	F	F			

■

2 Logical Equivalence

Definition 2.1. Two statement forms are called **logically equivalent** if, and only if, they have identical truth values for each possible substitution of statements for their statement variables.

Note: The *logical equivalence* of statement forms P and Q is denoted by writing

$$P \equiv Q.$$

Testing Whether Two Statement Forms P and Q Are Logically Equivalent

- Construct a truth table with one column for the truth values of P and another column for the truth values of Q.
- Check each combination of truth values of the statement variables to see whether the truth value of P is the same as the truth value of Q.
  - If in each row the truth value of P is the same as the truth value of Q, then P and Q are logically equivalent.
  - If in some row P has a different truth value from Q, then P and Q are not logically equivalent.

Example 2.1. (Double negation) Show that $p \equiv \sim (\sim p)$.
Solution:
Truth Table:

p	$\sim p$	$\sim (\sim p)$
T	F	T
F	T	F

Since the truth values are the same for all two rows in the truth table, then $p \equiv \sim (\sim p)$. ■

Example 2.2. Determine whether the statement forms

$$\sim (p \wedge q) \qquad \text{and} \qquad \sim p \wedge \sim q$$

are logically equivalent.
Solution:

2.1 Logical Equivalence: De Morgan's Laws

De Morgan's Laws

$$\sim (p \wedge q) \equiv \sim p \vee \sim q$$
$$\sim (p \vee q) \equiv \sim p \wedge \sim q$$

I.e., the De Morgan's Laws tell us the followings.

- The negation of an *and* statement is logically equivalent to the *or* statement in which each component is negated.
- The negation of an *or* statement is logically equivalent to the *and* statement in which each component is negated.

Example 2.3. Write negations for each of the following statements:

(i) John is 6 feet tall and he weighs at least 200 pounds.

(ii) The bus was late or Tom's watch was slow.

Solution:

(i)

(ii)

■

Example 2.4. Use De Morgan's laws to write the negation of

$$-1 < x \leq 4.$$

Solution:

2.2 Tautologies & Contradiction

Definition 2.2 (Tautologies & Contradiction)**.** :

- A **tautology** is a statement form that is always true regardless of the truth values of the individual statements substituted for its statement variables.

  ⇒ A statement whose form is a tautology is a *tautological statement*.

- A **contradiction** is a statement form that is always false regardless of the truth values of the individual statements substituted for its statement variables.

  ⇒ A statement whose form is a contradiction is called a *contradictory statement*.

Example 2.5. Show that

- $p \vee \sim p$ is a tautology,and
- $p \wedge \sim p$ is a contradiction.

Solution:

Example 2.6. Let t be a tautology and c be a contradiction. Show that

$$p \ \wedge t \equiv p \qquad \text{and} \qquad \mathrm{p} \ \wedge \mathrm{c} \equiv \mathrm{c}.$$

Solution:

Summary of Logical Equivalences

Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction c, the following logical equivalences hold.

	Logical Equivalent Laws		
1	Commutative laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2	Associative laws	$(p \wedge q) \wedge r \equiv q \wedge (p \wedge r)$	$(p \vee q) \vee r \equiv q \vee (p \vee r)$
3	Distributive laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4	Identity laws	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5	Negation laws	$p \vee \sim p \equiv \mathbf{t}$	$p \wedge \sim p \equiv \mathbf{c}$
6	Double negation law	$\sim (\sim p) \equiv p$	
7	Idempotent laws	$p \wedge p \equiv p$	$p \vee p \equiv p$
8	Universal bound laws	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9	De Morgan's laws	$\sim (p \wedge q) \equiv \sim p \vee \sim q$	$\sim (p \vee q) \equiv \sim p \wedge \sim q$
10	Absorption laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11	Negative of **t** and **c**	$\sim \mathbf{t} \equiv \mathbf{c}$	$\sim \mathbf{c} \equiv \mathbf{t}$

Example 2.7. Use laws of logical equivalences to show that

$$\sim (\sim p \wedge q) \wedge (p \vee q) \equiv p.$$

Solution:

3 Conditional & Biconditional Statements

Definition 3.1. Let p and q be statements. The **conditional statement** $p \to q$ is the statement that is false only when *p is true and q is false*, otherwise it is true.

- p is called **hypothesis** (or **antecedent**).
- q is called the **conclusion**(or **consequent**).
- The connective "$\to$" is called the **conditional connective**.

Truth table:

p	q	$p \to q$	
T	T	T	
T	F	F	
F	T	T	←-- **vacuously true**
F	F	T	←-- **vacuously true**

A conditional statement that is true due to the fact that its hypothesis is false (the last two cases) is also called **vacuously true** or **true by default**.
Note: The **implication** $\boxed{p \to q}$ is also called:

- "the conditional of q by p"
- "If p then q"
- "p implies q"
- " p only if q"
- "p is a **sufficient** condition for q"
- "q is a **necessary** condition for p."

Example 3.1. Rewrite the following statement in the form "If A then B":

Attending every lecture is a sufficient condition for a student to pass this class.

<u>Solution:</u>

Example 3.2. The statement

$$1 + 1 = 0, \text{ then } 2 = 1.$$

is **vacuously true** because the hypothesis is false.

Example 3.3. Consider the following statement:

"If the student shows up every lecture, this student will not fail the class. "

- The above statement is false 'F' only when ....

- If the student *does not* come to every lecture, this statement is always *true*, regardless of whether the conclusion is true or false (fail or pass).

  $\rightsquigarrow$ This is because............

- I.e., the above statement is called ...., if the student does not come to every lecture.

Example 3.4. Determine whether or not $\boxed{p \to q \equiv \sim p \vee q}$.
Solution: Truth table:

p	q	$\sim p$	$p \to q$	$\sim p \vee q$
T	T			
T	F			
F	T			
F	F			

Example 3.5. 1. Show that $\boxed{\sim (p \to q) \equiv p \wedge \ \sim q}$.

2. Find the negation of the statement:
   "If my car is in the repair shop, then I cannot go to class."

Solution:

1. We use De Morgan's laws as follows.

2. Let p be " my car is in the repair shop," and q be "I cannot go to class." Then, the given statement is equivalent to

3.1 Contrapositive/Converse/Inverse of Conditional Statements

Definition 3.2. :

- The **contrapositive** of a conditional statement $p \to q$ is $\boxed{\sim q \to \sim p}$.
- The **converse** of a conditional statement $p \to q$ is $\boxed{q \to p}$.
- The **inverse** of a conditional statement $p \to q$ is $\boxed{\sim p \to \sim q}$

Note:

- $\boxed{p \to q \equiv \sim q \to \sim p}$
- A conditional statement and its converse are *not* logically equivalent:
- A conditional statement and its inverse are *not* logically equivalent:
- The converse and the inverse of a conditional statement are logically equivalent to each other:

.

Example 3.6. Find the converse, inverse, and the contrapositive of the implication:

" If today is Monday, then I have a test today."

<u>Solution:</u> Let p be "*today is Monday*" and q be " I have a test today."

- The **converse** of this statement $\boxed{q \to p}$:
- The **inverse** of this statement $\boxed{\sim p \to \sim q}$:
- The **contrapositive** of this statement $\boxed{\sim q \to \sim p}$:

Example 3.7. Use truth table to show the following logical equivalence:

$$(p \vee q) \to r \equiv (p \to r) \wedge (q \to r).$$

Solution: Truth table:

p	q	r	$p \vee q$	$p \to r$	$q \to r$	$(p \vee q) \to r$	$(p \to r) \wedge (q \to r)$
T	T	T					
T	T	F					
T	F	T					
T	F	F					
F	T	T					
F	T	F					
F	F	T					
F	F	F					

Example 3.8. Rewrite the following statement in *if-then* form in two ways, one of which is the contrapositive of the other.
"John will break the world's record for the mile run only if he runs the mile in under four minutes."
Solution:
Version 1:
If John does not run the mile in under four minutes, then he will not break the world's record.
Version 2:
If John breaks the world's record, then he will have run the mile in under four minutes.

Summary of Logical Equivalences

- $p \to q \equiv \sim p \vee q$
- $p \to q \equiv \sim q \to \sim p$
- $\sim (p \to q) \equiv p \wedge \sim q$
- $(p \vee q \to r) \equiv (p \to r) \wedge (q \to r)$

3.2 Biconditional Statements

Definition 3.3. The **biconditional** statement of p and q, denoted by $\boxed{p \leftrightarrow q}$, is the statement form that is:

- true when both p and q have the same truth values, and
- false p and q have opposite truth values.

Note: $\boxed{p \leftrightarrow q}$ is also read:

- "p if and only if q" or
- " p is a necessary and sufficient condition for q."

Truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

The order of connective operations

1. $\sim$
2. $\wedge$ and $\vee$ in any order (coequal)
3. $\rightarrow$ and $\leftrightarrow$ in any order (coequal)

Note: The order of operations can be overridden through the use of parentheses. E.g.

- $\sim p \wedge q = (\sim p) \wedge q$.
- $\sim (p \wedge q)$ is the negation of p and q.
- $p \wedge q \vee r$ is considered ambiguous (must be either $(p \wedge q) \vee r$ or $p \wedge (q \vee r)$).

Example 3.9. Construct a truth table for $p \vee \sim q \rightarrow \sim p$.
Solution: Note that the above statement means $(p \vee (\sim q)) \rightarrow (\sim p)$. Truth table:

p	q	$\sim p$	$\sim q$	$p \vee \sim q$	$p \vee \sim q \rightarrow \sim p$
T	T				
T	F				
F	T				
F	F				

Example 3.10. Show that the biconditional proposition of p and q is logically equivalent to the conjunction of the conditional propositions $p \to q$ and $q \to p$.

$$p \leftrightarrow q \;\equiv\; (p \to q) \land (q \to p)$$

Solution: