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Group Homework 5 & 6

Semester 2/2021 EE320 Introductory mathematical economics

Due date: April 7th 2022 (before midnight / B.E. moodle).

Q1. Given the production function $Q = f(K, L) = A[K^n + L^n]$ where A is the level of technology, K is capital and L is labor. Suppose that $n > 0$. Consider the following problem.

a. To ensure that the above production function exhibits a *decreasing return to scale technology*, what additional restrictions do one need to place on n?

b. Under the assumption used in (a), show that the production function satisfies the law of diminishing returns.

c. Calculate the marginal rate of technical substitution (MRTS) of labor (L) for capital (K).

d. Show that MRTS is a decreasing function in L. That is, as labor increases, the value of MRTS decreases.

e. Under the condition(s) assumed in (a), is the production function satisfied with the *global* concave property?

Suppose that $K(t) = \frac{1}{2}t^2 + 2t + 3$ and $L(t) = e^t + 3$, where $t \geq 0$ is the number of periods from now. Consider the following problem

f. Show that Q is increasing over time.

g. Compute $\frac{\frac{dQ}{Q}}{dt}$ when t = 0, i.e. growth of output in the initial period.

Q2. A monopolist faces the market demand given by $P = Q^{-c}$ where "c" is a parameter with positive value, "P" is the price per unit output and "Q" is the amount of output. Suppose that monopolist's production technology is given by $Q = K^{\frac{1}{3}}L^{\frac{2}{3}}$ where "K" and "L" are the level of capital used and the number of labor employed, respectively. Assume that the unit price of K and L are set equal to "r" and "w", respectively. Consider the following problems.

a. What type of the return to scale technology does the production function exhibit?

From now on, assume that c = ¼ . Consider the following problems.

b. Construct the profit function of the monopolist. (Hint: your profit function should be expressed in terms of K and L.)

c. The firm wants to maximize profit and seek for combination of the two factor inputs . Derive the demand for factor inputs, capital and labor.

d. How does the demand for labor vary with respect to w and r? Show your result by using partial derivative.

e. Confirm your answer with the second-order condition.

Q3. In a duopoly market, suppose that firms 1 and 2 face market demand, $p = 100 - (q_1 + q_2)$. Firm costs are $c_1 = 10q_1$ and $c_2 = q_2^2$.

(a) Calculate market price under the Cournot environment. How much is the profit that each firm yields in the equilibrium?

(b) Calculate the market price under the joint production decision, i.e. collusive equilibrium. How much is the profit that each firm yields in the equilibrium?

(c) Do you think that the collusive equilibrium will sustain? What would be the profit of firm 1 if firm 1 chooses to deviate from the collusive allocation?

Q4. Consider a simple two-market model where demand and supply for each market is given below. (Notationally, let's name the two markets as A and B, respectively.)

Market A:

Demand: $p_A = 10 - 2Q_A$

Supply: $p_A = 1 + Q_A$

Market B:

Demand: $p_B = 20 - Q_B$

Supply: $p_B = 2 + 2Q_B$

a. Derive the market equilibrium
b. Suppose the government imposes unit tax on both markets at the rate of $t_A$and t_B. Solve for the after-tax equilibrium as the function of t_A and t_B.
c. How much revenue can the government collect from the taxation?
d. Determine the level of t_A and t_B that maximizes government's revenue.

Q5. (Monopolist and the Optimal advertising)

Consider a linear demand equation faced by a monopolist.

$$Q = 2000 + 4\sqrt{A} - 20P,$$

where A is the dollar amount of the expenditure on advertisement, P is the unit price, and Q is the amount of quantity demanded by consumers. Both θ and β are strictly positive.

The demand equation is an extended version of the traditional one. The proposed function captures an idea that advertising can boost the total demand in the market as potential customers get more information about the product.

Suppose that the monopolist cost function is given by $C(Q, A) = c(Q) + A$ where $c(Q) = 2Q + 1000$

Consider the following problem.

a) Construct the profit function.
b) Determine the optimal pricing (P*) and optimal advertisement (A*) that maximize the profit of the monopolist.
c) Confirm the result with the second-order differential test, i.e. hessian-matrix method.

Q1. Given the production function $Q = f(K, L) = A[K^n + L^n]$ where A is the level of technology, K is capital and L is labor. Suppose that $n > 0$. Consider the following problem.

a. To ensure that the above production function exhibits a *decreasing return to scale technology*, what additional restrictions do one need to place on n?

$$\begin{aligned} Q &= A[K^n + L^n] \\ &= A[(tK)^n + (tL)^n] \\ &= t^n \cdot A \cdot (K^n + L^n) \\ &= t^n \cdot Q \end{aligned}$$

The condition is $2n < 1$ to exhibit a decreasing return to scale. #

b. Under the assumption used in (a), show that the production function satisfies the law of diminishing returns.

$[MP_K]$: $\dfrac{\partial f(K,L)}{\partial K} = AnK^{n-1}$

$\dfrac{\partial MP_K}{\partial K} = \underbrace{(n-1)}_{\ominus \text{ according to a.)}} AnK^{n-2} < 0$

due to law of diminishing marginal product.
As K increases, MP_K falls. #

$[MP_L]$: $\dfrac{\partial f(K,L)}{\partial L} = AnL^{n-1}$

$\dfrac{\partial MP_L}{\partial L} = (n-1)AnL^{n-2} < 0$

due to law of diminishing marginal product.
As L increases, MP_L falls. #

c. Calculate the marginal rate of technical substitution (MRTS) of labor (L) for capital (K).

$$\begin{aligned} MRTS = \frac{MP_L}{MP_K} &= \frac{AnL^{n-1}}{AnK^{n-1}} \\ &= \left(\frac{L}{K}\right)^{n-1} \end{aligned}$$ #

d. Show that MRTS is a decreasing function in L. That is, as labor increases, the value of MRTS decreases.

$$\frac{d\,MRTS}{dL} = n-1 \left(\frac{L}{K}\right)^{n-2} \cdot (1)$$

$2n < 1 \Rightarrow n < \frac{1}{2}$. so, $(n-1) < 0$ making MRTS falls as L rises. #

e. Under the condition(s) assumed in (a), is the production function satisfied with the *global* concave property?

$$Q = A[K^n + L^n]$$

F.O.C. :

$$\frac{\partial Q}{\partial K} = AnK^{n-1}$$

$$\frac{\partial Q}{\partial L} = AnL^{n-1}$$

$H = \begin{bmatrix} f_{KK} & f_{KL} \\ f_{LK} & f_{LL} \end{bmatrix}$ S.O.C. :

$$H = \begin{bmatrix} (n-1)AnK^{n-2} & 0 \\ 0 & (n-1)AnL^{n-2} \end{bmatrix}$$

$|H_1| = (n-1)AnK^{n-2}$; from d.) $n < \frac{1}{2}$. Hence, $|H_1| < 0 \quad \forall (K,L)$

$|H_2| = \underbrace{[(n-1)AnK^{n-2}]}_{\text{negative } (-)} \cdot \underbrace{[(n-1)AnL^{n-2}]}_{\text{negative } (-)} - (0+0) > 0 \quad \forall (K,L)$

$\therefore$ H is negative definite $\forall (K,L)$. Q is globally concave. #

Suppose that $K(t) = \frac{1}{2}t^2 + 2t + 3$ and $L(t) = e^t + 3$, where $t \geq 0$ is the number of periods from now. Consider the following problem

f. Show that Q is increasing over time.

'Total derivative'

$\frac{dc}{dt}$ $Q \leftarrow K \leftarrow t$ ($\partial Q/\partial K \times K'(t)$), $Q \leftarrow L \leftarrow t$ ($\frac{\partial Q}{\partial L} \times L'(t)$)

$$Q = f(K,L)\ ; \quad \frac{dQ}{dt} = \frac{\partial Q}{\partial K}K'(t) + \frac{\partial Q}{\partial L}L'(t)$$

$$= AnK^{n-1} \cdot \underbrace{dK}_{t+2} + AnL^{n-1} \cdot \underbrace{dL}_{te^{t-1}}$$

$$= AnK^{n-1}(t+2) + AnL^{n-1}(e^t)$$

$\therefore$ Q is increasing over time. #

g. Compute $\frac{\frac{dQ}{Q}}{dt}$ when t = 0, i.e. growth of output in the initial period.

$\left.\frac{dQ}{dt}\right|_{t=0} \rightarrow t=0;$

$K(t) = \frac{1}{2}(0^2) + 2(0) + 3 = 3;$

$L(t) = e^0 + 3 = 4$

$$\left.\frac{dQ}{dt}\right|_{t=0} = AnK^{n-1}(t+2) + AnL^{n-1}(te^{t-1})$$

$$= AnK^{n-1}(0+2) + AnL^{n-1}(0e^{0-1})$$

$$= An(3)^{n-1} \cdot 2 + An4^{n-1} \cdot (0)$$

$$= 2An(3)^{n-1}$$ #

Q2. A monopolist faces the market demand given by $P = Q^{-c}$ where "c" is a parameter with positive value, "P" is the price per unit output and "Q" is the amount of output. Suppose that monopolist's production technology is given by $Q = K^{\frac{1}{3}}L^{\frac{2}{3}}$ where "K" and "L" are the level of capital used and the number of labor employed, respectively. Assume that the unit price of K and L are set equal to "r" and "w", respectively. Consider the following problems.

a. What type of the return to scale technology does the production function exhibit?

$$Q = K^{\alpha} \cdot L^{\beta} \quad ; \quad Q = K^{\frac{1}{3}} L^{\frac{2}{3}}$$

$$\alpha + \beta = \frac{1}{3} + \frac{2}{3}$$

$$= \frac{3}{3} = 1 \qquad \therefore \text{ It is a constant return to scale.} \#$$

From now on, assume that c = ¼ . Consider the following problems.

b. Construct the profit function of the monopolist. (Hint: your profit function should be expressed in terms of K and L.)

$$\pi(K,L) = P \cdot Q - C(K,L)$$
$$= (Q^{-\frac{1}{4}})Q - rK - wL$$
$$= (K^{\frac{1}{3}} L^{\frac{2}{3}})^{\frac{3}{4}} - rK - wL$$
$$= K^{\frac{1}{4}} \cdot L^{\frac{1}{2}} - rK - wL \quad \#$$

c. The firm wants to maximize profit and seek for combination of the two factor inputs . Derive the demand for factor inputs, capital and labor.

$$\pi(K,L) = K^{\frac{1}{4}} L^{\frac{1}{2}} - rK - wL$$

F.O.C.

$$\frac{\partial \pi}{\partial K} = 0 \longrightarrow \frac{1}{4} K^{-\frac{3}{4}} L^{\frac{1}{2}} - r = 0$$

$$\frac{1}{4} K^{-\frac{3}{4}} L^{\frac{1}{2}} = r$$

$$\frac{L^{\frac{1}{2}}}{4K^{\frac{3}{4}}} = r$$

$$\frac{L^{\frac{1}{2}}}{K^{\frac{3}{4}}} = 4r \quad \text{—— (1)}$$

$$\frac{\partial \pi}{\partial L} = 0 \longrightarrow \frac{1}{2} K^{\frac{1}{4}} L^{-\frac{1}{2}} - w = 0$$

$$\frac{1}{2} K^{\frac{1}{4}} L^{-\frac{1}{2}} = w$$

$$\frac{K^{\frac{1}{4}}}{2L^{\frac{1}{2}}} = w$$

$$\frac{K^{\frac{1}{4}}}{L^{\frac{1}{2}}} = 2w \quad \text{—— (2)}$$

① × ② ; $\dfrac{\cancel{L^{\frac{1}{2}}}}{K^{\frac{3}{4}}} \times \dfrac{K^{\frac{1}{4}}}{\cancel{L^{\frac{1}{2}}}} = 4r \times 2w$

$$\frac{K^{\frac{1}{4}}}{K^{\frac{3}{4}}} = 4r2w$$

$$K^{-\frac{1}{2}} = 4r2w$$

$$\left(\frac{1}{K^{\frac{1}{2}}}\right)^2 = (4r2w)^2$$

$$\frac{1}{K} = 16r^2 4w^2$$

$$\boxed{\frac{1}{64r^2w^2} = K}$$ sub into ② ;

$$\frac{\left(\frac{1}{64r^2w^2}\right)^{\frac{1}{4}}}{L^{\frac{1}{2}}} = 2w$$

$$\left(\frac{1^{\frac{1}{4}}}{64^{\frac{1}{4}} r^{\frac{1}{2}} w^{\frac{1}{2}} 2w}\right)^2 = \left(L^{\frac{1}{2}}\right)^2$$

$$\frac{1}{64^{\frac{1}{2}} r^1 w^1 4w^2} = L$$

$$\boxed{\frac{1}{32rw^3} = L}$$

Demand: $Q = \left(K^{\frac{1}{3}} L^{\frac{2}{3}}\right)$

$$= \left(\frac{1}{64r^2w^2}\right)^{\frac{1}{3}} \left(\frac{1}{32rw^3}\right)^{\frac{2}{3}}$$

$$= \frac{1}{64^{\frac{1}{3}} r^{\frac{2}{3}} w^{\frac{2}{3}}} \cdot \frac{1}{32^{\frac{2}{3}} r^{\frac{2}{3}} w^2}$$

$$= \frac{1}{4 r^{\frac{2}{3}} w^{\frac{2}{3}}} \cdot \frac{1}{\sqrt[3]{32^2}\, r^{\frac{2}{3}} w^2}$$

$$= \frac{1}{4\sqrt[3]{32^2}\, r^{\frac{4}{3}} w^{\frac{8}{3}}}$$ #

d. How does the demand for labor vary with respect to w and r? Show your result by using partial derivative.

$$L^* = \frac{1}{32rw^3} \rightarrow \frac{r^{-1}w^{-3}}{32}$$

$$\frac{\partial L^*}{\partial w} = \frac{-3}{32rw^4} \quad \#$$

When the unit price of labor rises by 1 unit, the number of labor employed will decrease by $\frac{3}{32rw^4}$.

$$\frac{\partial L^*}{\partial r} = \frac{-1}{32r^2w^3} \quad \#$$

When the unit price of capital rises by 1 unit, the number of labor employed will decrease by $\frac{1}{32r^2w^3}$.

e. Confirm your answer with the second-order condition.

F.O.C.

$$[K]: \frac{1}{4}K^{-\frac{3}{4}}L^{\frac{1}{2}} - r = 0$$

$$[L]: \frac{1}{2}K^{\frac{1}{4}}L^{-\frac{1}{2}} - w = 0$$

S.O.C.

$$H = \begin{bmatrix} \pi_{KK} & \pi_{KL} \\ \pi_{LK} & \pi_{LL} \end{bmatrix} = \begin{bmatrix} -\frac{3}{16}K^{-\frac{7}{4}}L^{\frac{1}{2}} & \frac{1}{8}K^{-\frac{3}{4}}L^{-\frac{1}{2}} \\ \frac{1}{8}K^{-\frac{3}{4}}L^{-\frac{1}{2}} & -\frac{1}{4}K^{\frac{1}{4}}L^{-\frac{3}{2}} \end{bmatrix}$$

$$|H_1| = -\frac{3}{16}K^{-\frac{7}{4}}L^{\frac{1}{2}} < 0 \rightarrow \forall (K,L)$$

$$|H_2| = \left(-\frac{3}{16}K^{-\frac{7}{4}}L^{\frac{1}{2}}\right) \times \left(-\frac{1}{4}K^{\frac{1}{4}}L^{-\frac{3}{2}}\right) - \left(\frac{1}{8}K^{-\frac{3}{4}}L^{-\frac{1}{2}}\right)^2 > 0 \rightarrow \forall (K,L)$$

$\therefore$ H is negative definite $\forall (K,L)$

π is globally concave.

So, (K^*, L^*) is global solution to true profit max input bundle. #

Q3. In a duopoly market, suppose that firms 1 and 2 face market demand, $p = 100 - (q_1 + q_2)$. Firm costs are $c_1 = 10q_1$ and $c_2 = q_2^2$.

(a) Calculate market price under the Cournot environment. How much is the profit that each firm yields in the equilibrium?

step 1

Firm 1

$$\pi^1 = p \cdot q_1 - c_1$$
$$= [100 - (q_1+q_2)]q_1 - 10q_1$$
$$= 100q_1 - q_1^2 - q_1q_2 - 10q_1$$

$$\frac{\partial \pi^1}{\partial q_1} = 0 \rightarrow 100 - 2q_1 - q_2 - 10 = 0$$
$$90 - 2q_1 - q_2 = 0$$
$$\Rightarrow q_1 = 45 - \frac{q_2}{2}$$

Firm 2

$$\pi_2 = p \cdot q_2 - c_2$$
$$= [100 - (q_1+q_2)]q_2 - q_2^2$$
$$= 100q_2 - q_1q_2 - q_2^2 - q_2^2$$

$$\frac{\partial \pi_2}{\partial q_2} = 0 \rightarrow 100 - q_1 - 2q_2 - 2q_2 = 0$$
$$100 - q_1 - 4q_2 = 0$$
$$\Rightarrow q_2 = 25 - \frac{q_1}{4}$$

step 2

$$q_1 = 45 - \frac{\left(25 - \frac{q_1}{4}\right)}{2}$$

$$2(q_1 - 45) = -25 + \frac{q_1}{4}$$
$$2q_1 - 90 = -25 + \frac{q_1}{4}$$
$$2q_1 - \frac{q_1}{4} = 90 - 25$$
$$\frac{7q_1}{4} = 65$$
$$q_1 = \frac{260}{7} \text{ units} \longrightarrow q_2 = 25 - \frac{\left(\frac{260}{7}\right)}{4} = \frac{110}{7} \text{ units}$$

$$\pi^1 = [100 - (q_1+q_2)]q_1 - 10q_1$$
$$= \left[100 - \left(\frac{260}{7} + \frac{110}{7}\right)\right]\frac{260}{7} - 10\left(\frac{260}{7}\right)$$
$$= 1{,}379.59 \; \#$$

$$\pi^2 = [100 - (q_1+q_2)]q_2 - q_2^2$$
$$= \left[100 - \left(\frac{260}{7} + \frac{110}{7}\right)\right]\frac{110}{7} - \left(\frac{110}{7}\right)^2$$
$$= 493.87 \; \#$$

(b) Calculate the market price under the joint production decision, i.e. collusive equilibrium. How much is the profit that each firm yields in the equilibrium?

$$P = 100 - (q_1+q_2) \quad \begin{cases} \underline{\underline{1}} & C_1 = 10q_1 \\ \underline{\underline{2}} & C_2 = q_2^2 \end{cases}$$

$$\begin{aligned} [Q_1, Q_2]\ \pi^c &= \pi^1 + \pi^2 \\ &= (P\cdot q_1 - C_1) + (P\cdot q_2 - C_2) \\ &= P(q_1+q_2) - 10q_1 - q_2^2 \\ &= [100-(q_1+q_2)](q_1+q_2) - 10q_1 - q_2^2 \\ &= [100(q_1+q_2) - (q_1+q_2)^2] - 10q_1 - q_2^2 \\ &= 100q_1 + 100q_2 - q_1^2 - 2q_1q_2 - q_2^2 - 10q_1 - q_2^2 \\ &= 90q_1 + 100q_2 - q_1^2 - 2q_1q_2 - 2q_2^2 \end{aligned}$$

Firm 1: $\frac{\partial \pi^c}{\partial q_1} = 90 - 2q_1 - 2q_2 = 0 \Rightarrow 90 = 2q_1 + 2q_2$

$45 = q_1 + q_2$ — 1

Firm 2: $\frac{\partial \pi^c}{\partial q_2} = 100 - 2q_1 - 4q_2 = 0 \Rightarrow 100 = 2q_1 + 4q_2$

$50 = q_1 + 2q_2$ — 2

2 − 1; $\boxed{q_2 = 5}$. Then sub into 1: $q_1 + 5 = 45$

$\boxed{q_1 = 40}$

$$\Rightarrow P = 100 - (q_1+q_2) = 100 - (40+5) = 55$$

$$\begin{aligned} \pi^1 &= P\cdot q_1 - C_1 \\ &= 55(40) - 10(40) = 1{,}800 \ \# \end{aligned}$$

$$\begin{aligned} \pi^2 &= P\cdot q_2 - C_2 \\ &= 55(5) - (5^2) = 250 \ \# \end{aligned}$$

(c) Do you think that the collusive equilibrium will sustain? What would be the profit of firm 1 if firm 1 chooses to $\boxed{\text{deviate}}$ from the collusive allocation?

Individual profit:	Collusive profit:
price = 47.14	price = 55
q_1 = 37.14 units	q_1 = 40 units
q_2 = 15.71 units	q_2 = 5 units
π^1 = 1,379.59	π^1 = 1,800
π^2 = 493.87	π^2 = 250

∴ As the price rises with the collusive allocation, the quantity sold from firm 1 increases; while, the quantity sold from firm 2 falls. Hence, it suggests that firm 1 will be more sustain under collusive equilibrium. #

∴ If firm 1 chooses to deviate from collusive allocation, it will receive profit of 1379.59. Thus, it foregoes additional of 1,800 − 1379.59 = 420.41 from taking the collusive allocation. #

Q4. Consider a simple two-market model where demand and supply for each market is given below. (Notationally, let's name the two markets as A and B, respectively.)

Market A:

Demand: $p_A^D = 10 - 2Q_A$

Supply: $p_A^S = 1 + Q_A$

Market B:

Demand: $p_B^D = 20 - Q_B$

Supply: $p_B^S = 2 + 2Q_B$

a. Derive the market equilibrium

Mkt. A: Demand = Supply

$$10 - 2Q_A = 1 + Q_A$$
$$9 = 3Q_A$$
$$Q_A^* = 3 \longrightarrow P_A^* = 4 \ \#$$

Mkt. B: Demand = Supply

$$20 - Q_B = 2 + 2Q_B$$
$$18 = 3Q_B$$
$$Q_B^* = 6 \longrightarrow P_B^* = 14 \ \#$$

b. Suppose the government imposes unit tax on both markets at the rate of t_A and t_B. Solve for the after-tax equilibrium as the function of t_A and t_B.

(A): $P_A^S = P_A^D - t_A$

$$1 + Q_A = 10 - 2Q_A - t_A$$
$$3Q_A = 9 - t_A$$

After-tax $Q_A = 3 - \frac{t_A}{3}$ #

$$P_A^S = 1 + 3 - \frac{t_A}{3} = 4 - \frac{t_A}{3} \ \#$$
$$P_A^D = 10 - 2\left(3 - \frac{t_A}{3}\right) = 4 + \frac{2t_A}{3} \ \#$$

(B): $P_B^S = P_B^D - t_B$

$$2 + 2Q_B = 20 - Q_B - t_B$$
$$3Q_B = 18 - t_B$$

After-tax $Q_B = 6 - \frac{t_B}{3}$ #

$$P_B^S = 2 + 2\left(6 - \frac{t_B}{3}\right) = 14 - \frac{2t_B}{3} \ \#$$
$$P_B^D = 20 - \left(6 - \frac{t_B}{3}\right) = 14 + \frac{t_B}{3} \ \#$$

c. How much revenue can the government collect from the taxation?

(R) Tax revenue = (unit tax) x (quantity after tax)

$$= Q_A \cdot t_A + Q_B \cdot t_B$$
$$= \left(3 - \frac{t_A}{3}\right)t_A + \left(6 - \frac{t_B}{3}\right)t_B$$
$$= 3t_A - \frac{t_A^2}{3} + 6t_B - \frac{t_B^2}{3} \ \#$$

d. Determine the level of t_A and t_B that maximizes government's revenue.

$$\max t_A: \quad \frac{\partial R}{\partial t_A} = 3 - \frac{2t_A}{3} = 0 \Rightarrow t_A^* = 4.5 \ \#$$
$$\max t_B: \quad \frac{\partial R}{\partial t_B} = 6 - \frac{2t_B}{3} = 0 \Rightarrow t_B^* = 9 \ \#$$

Q5. (Monopolist and the Optimal advertising)

Consider a linear demand equation faced by a monopolist.

$$Q = 2000 + 4\sqrt{A} - 20P,$$

where A is the dollar amount of the expenditure on advertisement, P is the unit price, and Q is the amount of quantity demanded by consumers. Both θ and β are strictly positive.

The demand equation is an extended version of the traditional one. The proposed function captures an idea that advertising can boost the total demand in the market as potential customers get more information about the product.

Suppose that the monopolist cost function is given by $C(Q, A) = c(Q) + A$ where $c(Q) = 2Q + 1000$

Consider the following problem.

a) Construct the profit function.

$$\pi = P \cdot Q - c(Q) - A$$

sub in $Q = 2{,}000 + 4\sqrt{A} - 20P$;

$$= P(2{,}000 + 4\sqrt{A} - 20P) - 2Q - 1{,}000 - A$$

$$= 2{,}000P + 4P\cdot\sqrt{A} - 20P^2 - 4{,}000 - 8\sqrt{A} + 40P - 1{,}000 - A$$

$$= 2{,}040P + 4PA^{\frac{1}{2}} - 20P^2 - 5{,}000 - 8A^{\frac{1}{2}} - A \quad \#$$

b) Determine the optimal pricing (P*) and optimal advertisement (A*) that maximize the profit of the monopolist.

$[P]$: $2{,}040 + 4A^{\frac{1}{2}} - 40P = 0$

$$40P - 4A^{\frac{1}{2}} = 2040$$

$$P^* = \frac{2{,}040 + 4A^{\frac{1}{2}}}{40} \quad \#$$

$[A]$: $2PA^{-\frac{1}{2}} - 4A^{-\frac{1}{2}} - 1 = 0$

$$\frac{2P}{A^{\frac{1}{2}}} - \frac{4}{A^{\frac{1}{2}}} = 1$$

$$2P - 4 = A^{\frac{1}{2}}$$

$$4P^2 - 16 = A^* \quad \#$$

sub in A^*: $P^* = \dfrac{2040 + 4(4P^2 - 16)^{\frac{1}{2}}}{40}$

$$= \frac{2{,}040 + 4(2P - 4)}{40}$$

$$= \frac{2040 + 8P - 16}{40}$$

$$40P = 2{,}040 + 8P - 16$$

$$32P = 2{,}024$$

$P^* = 63.25$ # sub into A^*;

$$A^* = 4(63.25)^2 - 16$$

$$= 15{,}986.25 \quad \#$$

c) Confirm the result with the second-order differential test, i.e. hessian-matrix method.

$$H = \begin{bmatrix} \pi_{PP} & \pi_{PA} \\ \pi_{AP} & \pi_{AA} \end{bmatrix} = \begin{bmatrix} -40 & 2A^{-\frac{1}{2}} \\ 2A^{-\frac{1}{2}} & A^{-\frac{3}{2}}(2-P) \end{bmatrix}$$

$$|H_1| = -40 < 0 \rightarrow \forall (P, A)$$

$$|H_2| = -40\left(A^{-\frac{3}{2}}(2-P)\right) - \left(2A^{-\frac{1}{2}}\right)^2 < 0 \rightarrow \forall (P, A)$$

H is negative definite $\forall (P, A)$. π is globally concave $\Rightarrow$ P^*, A^* are our global solution. The profit maximization bundle. #