

Derivation of OLS estimators

The OLS estimators of the three variable linear regression model

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + u_i$$

Differentiating the equation

$$\sum \hat{u}_i^2 = \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_{2i} - \hat{\beta}_3 X_{3i})^2$$

Partially with respect to the three unknowns and setting the resulting equations to zero, we obtain

$$\frac{\partial \sum \hat{u}_i^2}{\partial \hat{\beta}_1} = 2 \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_{2i} - \hat{\beta}_3 X_{3i})(-1) = 0$$

$$\frac{\partial \sum \hat{u}_i^2}{\partial \hat{\beta}_2} = 2 \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_{2i} - \hat{\beta}_3 X_{3i})(-X_{2i}) = 0$$

$$\frac{\partial \sum \hat{u}_i^2}{\partial \hat{\beta}_3} = 2 \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_{2i} - \hat{\beta}_3 X_{3i})(-X_{3i}) = 0$$

Simplifying these, we obtain

$$\begin{aligned}\bar{Y} &= \hat{\beta}_1 + \hat{\beta}_2 \bar{X}_2 + \hat{\beta}_3 \bar{X}_3 \\ \sum Y_i X_{2i} &= \hat{\beta}_1 \sum X_{2i} + \hat{\beta}_2 \sum X_{2i}^2 + \hat{\beta}_3 \sum X_{2i} X_{3i} \\ \sum Y_i X_{3i} &= \hat{\beta}_1 \sum X_{3i} + \hat{\beta}_2 \sum X_{2i} X_{3i} + \hat{\beta}_3 \sum X_{3i}^2\end{aligned}$$

$$\begin{aligned}\hat{\beta}_1 &= \bar{Y} - \hat{\beta}_2 \bar{X}_2 - \hat{\beta}_3 \bar{X}_3 \\ \hat{\beta}_2 &= \frac{(\sum y_i x_{2i})(\sum x_{3i}^2) - (\sum y_i x_{3i})(\sum x_{2i} x_{3i})}{(\sum x_{2i}^2)(\sum x_{3i}^2) - (\sum x_{2i} x_{3i})^2} \\ \hat{\beta}_3 &= \frac{(\sum y_i x_{3i})(\sum x_{2i}^2) - (\sum y_i x_{2i})(\sum x_{2i} x_{3i})}{(\sum x_{2i}^2)(\sum x_{3i}^2) - (\sum x_{2i} x_{3i})^2}\end{aligned}$$

The OLS estimators of the k- variable linear regression model

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \dots + \beta_k X_{ki} + u_i$$

Differentiating the equation

$$\sum \hat{u}_i^2 = \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_{2i} - \hat{\beta}_3 X_{3i} - \dots - \hat{\beta}_k X_{ki})^2$$

Differentiating this expression partially with respect to each of the k unknowns and setting the resulting equations to zero, and rearranging, we obtain the following k normal equations in the k unknowns.

$$\begin{aligned} \sum Y_i &= n\hat{\beta}_1 + \hat{\beta}_2 \sum X_{2i} + \hat{\beta}_3 \sum X_{3i} + \dots + \hat{\beta}_k \sum X_{ki} \\ \sum Y_i X_{2i} &= \hat{\beta}_1 \sum X_{2i} + \hat{\beta}_2 \sum X_{2i}^2 + \hat{\beta}_3 \sum X_{2i} X_{3i} + \dots + \hat{\beta}_k \sum X_{2i} X_{ki} \\ &\dots\dots\dots \\ \sum Y_i X_{ki} &= \hat{\beta}_1 \sum X_{ki} + \hat{\beta}_2 \sum X_{2i} X_{ki} + \hat{\beta}_3 \sum X_{3i} X_{ki} + \dots + \hat{\beta}_k \sum X_{ki}^2 \end{aligned}$$