

Relations & Functions III

TU152: Fundamental Mathematics

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Definition (Composite Function)

Let $f : X \rightarrow \hat{Y}$ and $g : Y \rightarrow Z$ be functions with the property that the range of f is a subset of the domain of g , $\hat{Y} \subseteq Y$. Define a new function $g \circ f : X \rightarrow Z$ as follows:

$$(g \circ f)(x) = g(f(x)) \text{ for all } x \in X,$$

where

- $g \circ f$ is read “ g circle f ” and
- $g(f(x))$ is read “ g of f of x .”

The function $g \circ f$ is called the **composition** of f and g .

Remarks:

- Composition of functions is **not a commutative operation**.
For general functions f and g , $f \circ g$ need not necessarily equal $g \circ f$ (although the two may be equal).

Definition

For $f : X \rightarrow \hat{Y}$ and $g : Y \rightarrow Z$ with $\hat{Y} \not\subseteq Y$, but $\hat{Y} \cap Y \neq \emptyset$. We can still define a new function $g \circ f : \hat{X} \rightarrow Z$ as follows:

$$(g \circ f)(x) = g(f(x)) \text{ for all } x \in \hat{X} \subseteq X.$$

Notice that when $\hat{Y} \subseteq Y$, the domain $\hat{X}$ of $g \circ f$ will be the same as the domain X of f . However $\hat{Y} \not\subseteq Y$, but $\hat{Y} \cap Y \neq \emptyset$, $\hat{X}$ may be smaller than X .

Example: Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ be the successor function and let $g : \mathbb{Z} \rightarrow \mathbb{Z}$ be the squaring function. Then $f(n) = n + 1$ for all $n \in \mathbb{Z}$ and $g(n) = n^2$ for all $n \in \mathbb{Z}$.

- Find the compositions $g \circ f$ and $f \circ g$.
- Is $g \circ f = f \circ g$? Explain.

Example: Let $X = \{1, 2, 3\}$, $\hat{Y} = \{a, b, c, d\}$, $Y = \{a, b, c, d, e\}$, and $Z = \{x, y, z\}$.

Define functions $f : X \rightarrow \hat{Y}$ and $g : Y \rightarrow Z$ by

- $f(1) = c, f(2) = b, f(3) = a,$
- $g(a) = y, g(b) = y, g(c) = z, g(d) = z, g(e) = z.$

Draw the arrow diagrams for f , g , and $g \circ f$.

Determine the range for each of f , g , and $g \circ f$.

Example: Let $f(x) = \frac{2}{\sqrt{x-1}}$ and $g(x) = x^2 + 3$. Find $g \circ f$ and $f \circ g$. Find $(g \circ f)(2)$.

Example (Composition with the Identity Function):

Let $X = \{a, b, c, d\}$ and $Y = \{u, v, w\}$, and suppose $f : X \rightarrow Y$ is given by

$$f(a) = u, \quad f(b) = v, \quad f(c) = v, \quad f(d) = u.$$

Let $I_X : X \rightarrow X$ and $I_Y : Y \rightarrow Y$ with $I_X(x) = x$ and $I_Y(y) = y$ be identity functions.

Find $f \circ I_X$ and $I_Y \circ f$.

Composition with an Identity Function

Theorem: Composition with an Identity Function

If f is a function from a set X to a set Y , and I_X is the identity function on X , and I_Y is the identity function on Y , i.e. ,

$$I_X(x) = x \quad \forall x \in X, \quad I_Y(y) = y \quad \forall y \in Y$$

then

$$f \circ I_X = f \quad \text{and} \quad I_Y \circ f = f.$$

Let f be a function from a set X to a set Y , and suppose f has an inverse function f^{-1} . Recall that f^{-1} is the function from Y to X with the property that

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y.$$

What happens when f is composed with f^{-1} ? Or when f^{-1} is composed with f ?

Example: Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Define $f : X \rightarrow Y$ by

$$f(a) = z, \quad f(b) = x, \quad f(c) = y.$$

- 1 Draw the arrow diagram for f .
- 2 Show that the inverse function f^{-1} exists for the function f .
- 3 Find f^{-1} and draw the arrow diagram for f^{-1} .
- 4 Find and draw the arrow diagrams for $f^{-1} \circ f$ and $f \circ f^{-1}$. Compare with I_X and I_Y .

Example (continued):

Composition of a Function with Its Inverse

Theorem: Composition of a Function with Its Inverse

If $f : X \rightarrow Y$ is a one-to-one and onto function with inverse function $f^{-1} : Y \rightarrow X$, then

$$f^{-1} \circ f = I_X \quad \text{and} \quad f \circ f^{-1} = I_Y.$$

Example: Show that the following statement is true.

Theorem: The composition of two injective(one-to-one) functions is also injective

If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are both **one-to-one** functions, then $g \circ f$ is **one-to-one**.

Example: Show that the following statement is true.

Theorem: The composition of two surjective(onto) functions is also surjective(onto)

If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are both **onto** functions, then $g \circ f$ is **onto**.