Estimation Methods

Nonparametric Estimation Methods

- No assumption of distribution
- i.e. Linear Programming

Parametric Estimation Methods

- Assume distribution
  - Least Squares Methods (LS)
  - Maximum Likelihood (ML)
  - Generalized Method of Moments (GMM)

Least Squares Estimation Methods

Ordinary Least Squares (OLS)

Generalized Least Squares (GLS)

Feasible Generalized Least Squares (FGLS)

- Weighted Least Squares (WLS)

  Heteroscedasticity

- Cochrane-Orcutt Technique

  Autocorrelation

Other Least Squares Methods

- Nonlinear Least Squares (NLS)
- System Equation Estimation Methods

OLS Matrix Approach

Scalar Notation

$$y_i = \beta_1 + \beta_2 x_{2i} + \beta_3 x_{3i} + \cdots + \beta_k x_{ki} + u_i$$

$$where \quad i = 1, 2, 3, \ldots, n$$

$$\begin{aligned}
y_1 &= \beta_1 + \beta_2 x_{21} + \beta_3 x_{31} + \cdots + \beta_k x_{k1} + u_1 \\
y_2 &= \beta_1 + \beta_2 x_{22} + \beta_3 x_{32} + \cdots + \beta_k x_{k2} + u_2 \\
&\vdots \\
y_n &= \beta_1 + \beta_2 x_{2n} + \beta_3 x_{3n} + \cdots + \beta_k x_{kn} + u_n
\end{aligned}$$

$$\underset{n\times 1}{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}} = \underset{n\times k}{\begin{bmatrix} 1 & x_{21} & x_{31} & \cdots & x_{k1} \\ 1 & x_{22} & x_{32} & \cdots & x_{k2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{2n} & x_{3n} & \cdots & x_{kn} \end{bmatrix}} \cdot \underset{k\times 1}{\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{bmatrix}} + \underset{n\times 1}{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}}$$

OLS Matrix Approach

$$\underset{n\times 1}{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}} = \underset{n\times k}{\begin{bmatrix} 1 & x_{21} & x_{31} & \cdots & x_{k1} \\ 1 & x_{22} & x_{32} & \cdots & x_{k2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{2n} & x_{3n} & \cdots & x_{kn} \end{bmatrix}} \cdot \underset{k\times 1}{\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{bmatrix}} + \underset{n\times 1}{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}}$$

Matrix Notation

$$\underset{n\times 1}{Y} = \underset{n\times k}{X}\ \underset{k\times 1}{\beta} + \underset{n\times 1}{u}$$

Least Squares

$$\hat{u}'\hat{u} = \begin{bmatrix} \hat{u}_1 & \hat{u}_2 & \cdots & \hat{u}_n \end{bmatrix} \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_n \end{bmatrix} = \sum_{i=1}^{n} \hat{u}_i^2$$

OLS Matrix Approach

From
$$Y = X\beta + u$$
$$\hat{u} = Y - X\hat{\beta}$$

Least Squares
$$\hat{u}'\hat{u} = \left(Y - X\hat{\beta}\right)'\left(Y - X\hat{\beta}\right)$$
$$= Y'Y - 2\hat{\beta}'X'Y + \hat{\beta}'X'X\hat{\beta}$$
$$\frac{\partial \hat{u}'\hat{u}}{\partial \hat{\beta}} = -2X'Y + 2X'X\hat{\beta} = 0$$
$$\underset{k\times 1}{\hat{\beta}} = (\underset{k\times k}{X'X})^{-1}\underset{k\times n}{X'}\underset{n\times 1}{Y}$$

Assume normal distribution: $u \sim N(0, \sigma^2 I)$

OLS Matrix Approach

Variance-Covariance Matrix

$$uu' = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} = \begin{bmatrix} u_1u_1 & u_1u_2 & \cdots & u_1u_n \\ u_2u_1 & u_2u_2 & \cdots & u_2u_n \\ \vdots & \vdots & \ddots & \vdots \\ u_nu_1 & u_nu_2 & \cdots & u_nu_n \end{bmatrix}$$

$$E(uu') = \begin{bmatrix} E(u_1^2) & E(u_1u_2) & \cdots & E(u_1u_n) \\ E(u_2u_1) & E(u_2^2) & \cdots & E(u_2u_n) \\ \vdots & \vdots & \ddots & \vdots \\ E(u_nu_1) & E(u_nu_2) & \cdots & E(u_n^2) \end{bmatrix}$$

OLS Matrix Approach

Variance-Covariance Matrix

$$E(uu') = \underset{n\times n}{\Sigma} = \begin{bmatrix} E(u_1^2) & E(u_1u_2) & \cdots & E(u_1u_n) \\ E(u_2u_1) & E(u_2^2) & \cdots & E(u_2u_n) \\ \vdots & \vdots & \ddots & \vdots \\ E(u_nu_1) & E(u_nu_2) & \cdots & E(u_n^2) \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n3} & \cdots & \sigma_{\mathrm{n}}^2 \end{bmatrix}$$

Variance-Covariance Matrix

Under OLS Assumptions

$$\sum_{n\times n} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n3} & \cdots & \sigma_{\mathrm{n}}^2 \end{bmatrix} = \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix} = \sigma^2 I$$

Under Heteroscedasticity Problem

$$\sum_{n\times n} = \begin{bmatrix} \sigma_1^2 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_n^2 \end{bmatrix}$$

Weighted Least Squares (WLS)

Scalar

$$y_i = \beta_1 + \beta_2 x_i + u_i$$

$$\frac{y_i}{\hat{\sigma}_i} = \beta_1\left(\frac{1}{\hat{\sigma}_i}\right) + \beta_2\left(\frac{x_i}{\hat{\sigma}_i}\right) + \left(\frac{u_i}{\hat{\sigma}_i}\right)$$

$$y_i^* = \beta_1^* x_{0i}^* + \beta_2^* x_i^* + u_i^*$$

Matrix

$$\underset{k\times 1}{\hat{\beta}} = (\underset{k\times n}{X'}\,\underset{\substack{n\times n\\ k\times k}}{\hat{\Sigma}^{-1}}\,\underset{n\times k}{X})^{-1}\,\underset{k\times n}{X'}\,\underset{\substack{n\times n\\ k\times 1}}{\hat{\Sigma}^{-1}}\,\underset{n\times 1}{Y}$$

Autocorrelation: Cochrane-Orcutt

Scalar $$y_t = \beta_1 + \beta_2 x_t + u_t$$

$$u_t = \rho u_{t-1} + \varepsilon_t$$

1. Estimate model using OLS and obtain estimated u_t $\quad \hat{u}_t = \hat{\rho}\hat{u}_{t-1} + \upsilon_t$
2. Compute
3. $\left(y_t - \hat{\rho} y_{t-1}\right) = \beta_1 \left(1 - \hat{\rho}\right) + \beta_2 \left(x_t - \hat{\rho} x_{t-1}\right) + \varepsilon_t$

$$y_t^* = \beta_1^* + \beta_2^* x_t^* + \varepsilon_t^*$$

4. Iterative procedure to estimate ρ

Autocorrelation: Cochrane-Orcutt

Variance-Covariance Matrix $\underset{n\times n}{\Sigma} = \sigma^2 \underset{n\times n}{\Omega}$

where

$$\underset{\mathrm{n\times n}}{\Omega} = \begin{bmatrix} 1 & \rho & \rho^2 & \rho^3 & \cdots & \rho^{n-1} \\ \rho & 1 & \rho & \rho^2 & \cdots & \rho^{n-2} \\ \rho^2 & \rho & 1 & \rho & \cdots & \rho^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \cdots & \rho & 1 \end{bmatrix}$$

Matrix

$$\underset{k\times 1}{\hat{\beta}} = (\underset{k\times n}{X'} \underset{\substack{n\times n \\ k\times k}}{\hat{\Omega}^{-1}} \underset{n\times k}{X})^{-1} \underset{k\times n}{X'} \underset{\substack{n\times n \\ k\times 1}}{\hat{\Omega}^{-1}} \underset{n\times 1}{Y}$$

OLS vs GLS vs FGLS

OLS Estimation

$$\underset{n\times n}{\Sigma} = \sigma^2 \underset{n\times n}{I}$$

$$\underset{k\times 1}{\hat{\beta}} = (\underset{k\times k}{X'X})^{-1} \underset{k\times n}{X'} \underset{n\times 1}{Y}$$

WLS Estimation

$$\underset{n\times n}{\hat{\Sigma}} = \hat{\sigma}_i^2 \underset{n\times n}{I}$$

$$\underset{k\times 1}{\hat{\beta}} = (\underbrace{\underset{k\times n}{X'} \underset{n\times n}{\hat{\Sigma}}^{-1} \underset{n\times k}{X}}_{k\times k})^{-1} \underbrace{\underset{k\times n}{X'} \underset{n\times n}{\hat{\Sigma}}^{-1} \underset{n\times 1}{Y}}_{k\times 1}$$

Cochrane-Orcutt

$$\underset{n\times n}{\hat{\Sigma}} = \sigma^2 \underset{n\times n}{\hat{\Omega}}$$

$$\underset{k\times 1}{\hat{\beta}} = (\underbrace{\underset{k\times n}{X'} \underset{n\times n}{\hat{\Omega}}^{-1} \underset{n\times k}{X}}_{k\times k})^{-1} \underbrace{\underset{k\times n}{X'} \underset{n\times n}{\hat{\Omega}}^{-1} \underset{n\times 1}{Y}}_{k\times 1}$$

GLS Estimation

$$\underset{n\times n}{\Sigma} \text{ is known}$$

$$\underset{k\times 1}{\hat{\beta}} = (\underbrace{\underset{k\times n}{X'} \underset{n\times n}{\Sigma}^{-1} \underset{n\times k}{X}}_{k\times k})^{-1} \underbrace{\underset{k\times n}{X'} \underset{n\times n}{\Sigma}^{-1} \underset{n\times 1}{Y}}_{k\times 1}$$

FGLS Estimation

$$\underset{n\times n}{\Sigma} \text{ is not known}$$

$$\underset{k\times 1}{\hat{\beta}} = (\underbrace{\underset{k\times n}{X'} \underset{n\times n}{\hat{\Sigma}}^{-1} \underset{n\times k}{X}}_{k\times k})^{-1} \underbrace{\underset{k\times n}{X'} \underset{n\times n}{\hat{\Sigma}}^{-1} \underset{n\times 1}{Y}}_{k\times 1}$$