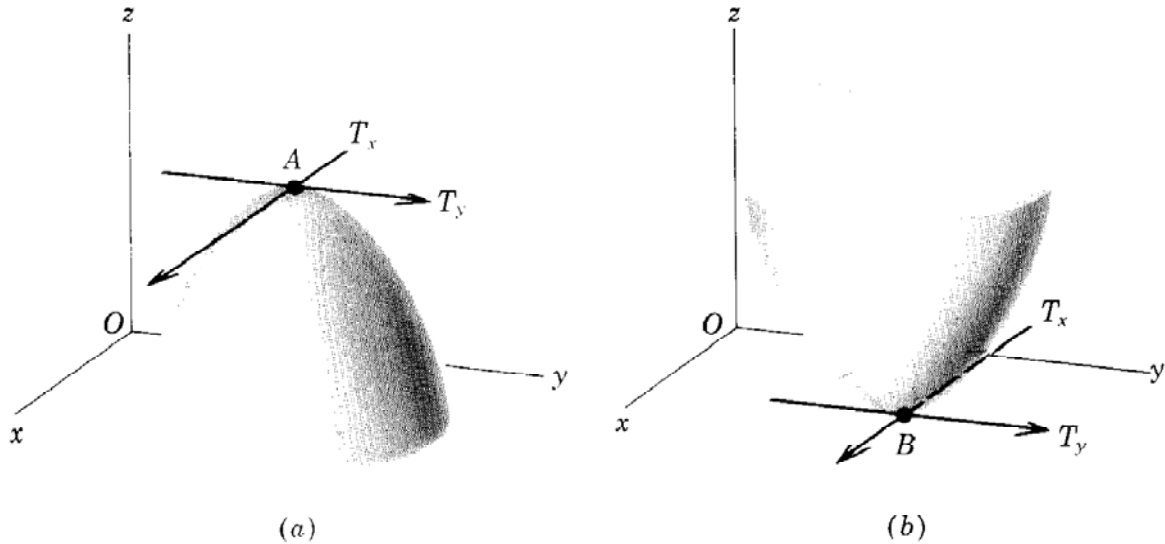


Chapter 6

Derivatives of More-Than-One Independent Variable Function

6.1 Extreme Values of a Function of Two Variables: First-Order Condition



For the function

$$z = f(x,y)$$

The first-order necessary condition for an extreme is $dz = 0$.

dz is a total differential.

$$dz = f_x dx + f_y dy$$

$$dz = 0 \text{ when } f_x = f_y = 0$$

Figure (a) is a maximum point. Figure (b) is a minimum point.

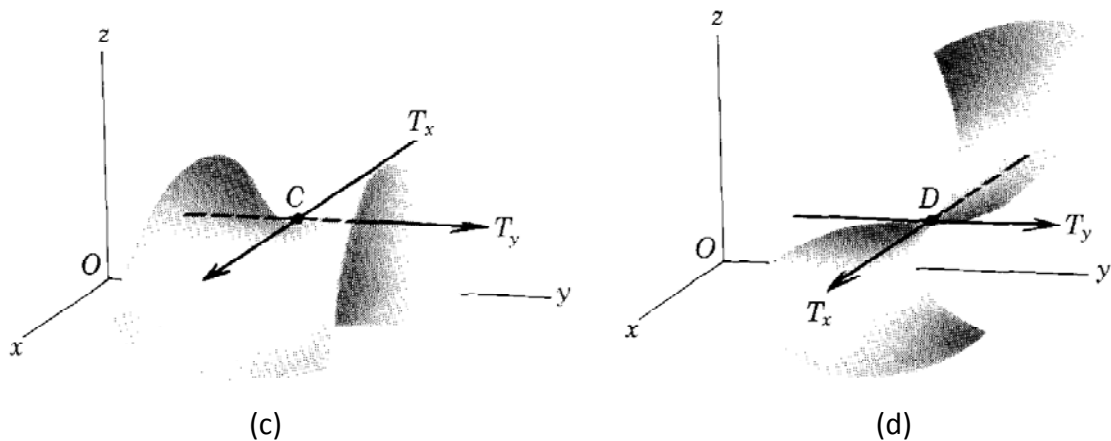


Figure (c) is a saddle point. Figure (d) is an inflection point.

6.2 Extreme Values of a Function of Two Variables: Second-Order Partial Derivatives

The function $z = f(x,y)$ can give rise to two first-order partial derivatives.

$$f_x \equiv \frac{\partial z}{\partial x} \quad \text{and} \quad f_y \equiv \frac{\partial z}{\partial y}$$

Second-order partial derivative

$$f_{xx} \equiv \frac{\partial}{\partial x}(f_x) \quad \text{or} \quad \frac{\partial^2 z}{\partial x^2} \equiv \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right)$$

The rate of change of f_x with respect to x , while y remains fixed.

$$f_{yy} \equiv \frac{\partial}{\partial y}(f_y) \quad \text{or} \quad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right)$$

The rate of change of f_y with respect to y , while x remains fixed.

f_x is also a function of y and f_y is also a function of x .

$$f_{xy} \equiv \frac{\partial^2 z}{\partial x \partial y} \equiv \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) \quad \text{and} \quad f_{yx} \equiv \frac{\partial^2 z}{\partial y \partial x} \equiv \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right)$$

The rate of change of one first-order partial derivative with respect to the “other” variable.

6.3 Differentials

dy and dx → differentials

Let $y = f(x)$

$dy = f'(x) dx$ → differential of y

6.4 Total Differential

$$y = f(A,B)$$

Total differential of y: $dy = \frac{\partial y}{\partial A} * dA + \frac{\partial y}{\partial B} * dB$

or $dy = f_A dA + f_B dB$

6.5 Total Derivative

$$z = f(x,y)$$

$$y = g(x)$$

$$dz = f_x dx + f_y dy$$

Total derivative of z with respect to x: $\frac{dz}{dx} = f_x + f_y \frac{dy}{dx}$

In general: If $z = f(x,y)$

and $x = g(w)$

$$y = h(w)$$

so $\frac{dz}{dw} = f_x \frac{dx}{dw} + f_y \frac{dy}{dw}$

6.6 Implicit Function

If $F(y, x_1, x_2, \dots, x_n) = 0$

The implicit function: $y = f(x_1, x_2, \dots, x_n)$

Implicit function theorem:

Let $F(y, x_1, x_2) = 0$

$$\text{and } F_y = \frac{\partial F}{\partial y}, F_1 = \frac{\partial F}{\partial x_1}, F_2 = \frac{\partial F}{\partial x_2}$$

Total differential: $F_y dy + F_1 dx_1 + F_2 dx_2 = 0$

If x_2 is constant or $dx_2 = 0$,

$$F_y dy + F_1 dx_1 = 0$$

so $\frac{dy}{dx_1} \Big|_{dx_2=0} = -\frac{F_1}{F_y}$

or $\frac{\partial y}{\partial x_1} = -\frac{F_1}{F_y}$

and when x_1 is constant; $\frac{\partial y}{\partial x_2} = -\frac{F_2}{F_y}$

In general:

$$\boxed{\frac{\partial y}{\partial x_i} = -\frac{F_i}{F_y}}$$

6.7 Economic Application

6.7.1 Partial Market Equilibrium

Product Market:

At equilibrium $Q_d = Q_s$

Demand equation: $Q_d = f(P) = a_1 + b_1P$

Supply equation: $Q_s = g(P) = a_2 + b_2P$

where $a_1 > 0, b_1 < 0$

$a_2 > 0, b_2 > 0$

and $a_1 b_2 > a_2 b_1$

In general:

$$Q_d = Q_s$$

$$Q_d = f(a_1, P)$$

$$Q_s = g(a_2, P)$$

Total differential:

$$dQ_d = dQ_s$$

$$dQ_d = f_{a_1} da_1 + f_P dP$$

$$dQ_s = g_{a_2} da_2 + g_P dP$$

$f_{a_1} > 0$, $f_P < 0$, $g_{a_2} > 0$ and $g_P > 0$

$$f_{a_1} da_1 + f_P dP = g_{a_2} da_2 + g_P dP$$

$$f_P dP - g_P dP = g_{a_2} da_2 - f_{a_1} da_1$$

$$dP(f_P - g_P) = g_{a_2} da_2 - f_{a_1} da_1$$

$$dP = \frac{g_{a_2} da_2 - f_{a_1} da_1}{f_P - g_P}$$

If $da_1 = 0$

$$\frac{dP}{da_2} \equiv \frac{\partial P}{\partial a_2} = \frac{g_{a_2}}{f_P - g_P} < 0 \quad (f_P < 0, g_P > 0 \text{ and } g_{a_2} > 0)$$

If $da_2 = 0$

$$\frac{dP}{da_1} \equiv \frac{\partial P}{\partial a_1} = - \frac{f_{a_1}}{f_P - g_P} > 0 \quad (f_{a_1} > 0, f_P < 0 \text{ and } g_P > 0)$$

6.7.1 Utility function

$$u = u(A, B)$$

$$du = u_A dA + u_B dB$$

$$\frac{dB}{dA} = - \frac{u_A}{u_B} = MRS_{AB}$$

$MRS_{AB} < 0 \rightarrow$ if A and B are both goods or A and B are both bads.

$MRS_{AB} > 0 \rightarrow$ if one of them is a good and another is a bad.

$MRS_{AB} = \infty \rightarrow$ if B is neutral, A is a good or a bad.

$MRS_{AB} = 0 \rightarrow$ if B is a good or a bad, A is neutral.

6.7.3 Multipliers in Macroeconomic Models

$$Y = C + I_0 + G_0$$

$$C = a + bY_d$$

$$Y_d = Y - T$$

$$T = tY$$

where I_0 and G_0 are autonomous variables.

$$0 < t < 1$$

$$0 < b < 1$$

$$a > 0$$

$$Y = a + bY - btY + I_0 + G_0$$

$$Y_e = \frac{a + I_0 + G_0}{1 - b(1-t)} = \frac{A}{u}$$

National income multipliers:

$$\frac{\partial Y_e}{\partial a} = \frac{\partial Y_e}{\partial I_0} = \frac{\partial Y_e}{\partial G_0} = \frac{1}{1 - b(1-t)}$$

$$\frac{\partial Y_e}{\partial b} = \frac{\partial Y_e}{\partial u} * \frac{\partial u}{\partial b}$$

$$= - \frac{A}{u^2} (-(1-t))$$

$$= \frac{A(1-t)}{u^2}$$

$$= \frac{(1-t)A}{[1-b(1-t)]^2} > 0$$

$$\begin{aligned} \frac{\partial Y_e}{\partial t} &= \frac{\partial Y_e}{\partial u} * \frac{\partial u}{\partial t} \\ &= -\frac{A}{u^2} (b) \\ &= \frac{-bA}{[1-b(1-t)]^2} < 0 \end{aligned}$$

6.7.4 Demand Elasticity

Demand function:

$$Q_{da} = f(P_a, P_b, Y)$$

$$\epsilon_p = \frac{\partial Q_{da}}{\partial P_a} * \frac{P_a}{Q_{da}}$$

$$\epsilon_{ab} = \frac{\partial Q_{da}}{\partial P_b} * \frac{P_b}{Q_{da}}$$

$$\epsilon_y = \frac{\partial Q_{da}}{\partial Y} * \frac{Y}{Q_{da}}$$

6.7.5 Production function

$$Q = f(L, K)$$

Cobb-Douglas production function:

$$Q = AL^\alpha K^\beta$$

where A is a constant and > 0 .

A \rightarrow coefficient of neutral technology

α and $\beta \rightarrow$ coefficients

$\rightarrow 0 < \alpha < 1$ and $0 < \beta < 1$

1. Marginal product:

$$MP_L = \frac{\partial Q}{\partial L} = \alpha AL^{\alpha-1}K^\beta$$

$$= \frac{\alpha AL^\alpha K^\beta}{L}$$

$$= \alpha \left(\frac{Q}{L} \right)$$

$$= \alpha(AP_L)$$

$$MP_K = \frac{\partial Q}{\partial K} = \beta AL^\alpha K^{\beta-1}$$

$$= \frac{\beta AL^\alpha K^\beta}{K}$$

$$= \beta \left(\frac{Q}{K} \right)$$

$$= \beta(AP_K)$$

2. $MP_L > 0$ and $MP_K > 0$ if L and $K > 0$

$$F_{LL} = \frac{\partial MP_L}{\partial L} \\ = \alpha(\alpha-1)AL^{\alpha-2}K^\beta \\ = \frac{\partial^2 Q}{\partial L^2} < 0; \quad (\alpha-1) < 0$$

$$F_{KK} = \frac{\partial MP_K}{\partial K} \\ = \beta(\beta-1)AL^\alpha K^{\beta-2}$$

$$= \frac{\partial^2 Q}{\partial K^2} < 0; \quad (\beta-1) < 0$$

MP_L and MP_K are monotonic decreasing functions.

3. MP_L and $MP_K \neq 0$ if L and $K \neq 0 \rightarrow$ this production function is a monotonic increasing function.

$$4. \quad \epsilon_L = \frac{\partial Q}{\partial L} * \frac{L}{Q} = \frac{MPL}{APL}$$

$$\epsilon_K = \frac{\partial Q}{\partial K} * \frac{K}{Q} = \frac{MPK}{APK}$$

For Cobb-Douglas production function

$$\epsilon_L = \alpha \left(\frac{Q}{L} \right) \left(\frac{L}{Q} \right) = \alpha$$

$$\epsilon_K = \beta \left(\frac{Q}{K} \right) \left(\frac{K}{Q} \right) = \beta$$

Let $R = (\alpha + \beta)$

If $R = 1 \rightarrow$ constant returns to scale

If $R < 1 \rightarrow$ decreasing returns to scale

If $R > 1 \rightarrow$ increasing returns to scale

Homogeneous function:

The Cobb-Douglas production function is a homogeneous function of degree $\alpha + \beta$.