7.3 Constrained optimization

Statement of the problem: Comparison

Constrain-free optimization	**Constraint optimization**
$\max\ (\min)_{x,y}\ f(x,y)$	$\max\ (\min)_{x,y}\ f(x,y)$
	such that
	$g(x,y) = c$

$(x,y) \in D_f$

"Entire domain of f that defines the objective f^n"

– possible combination of (x,y) that make $g(x,y)=c$

Statement of the problem: Comparison

Constrain-free optimization	**Constraint optimization**
$\max\ 200-(x-10)^2-(y-10)^2$	$\max\ 200-(x-10)^2-(y-10)^2$
	such that
	$x+y=10$

$(x=10, y=10)$

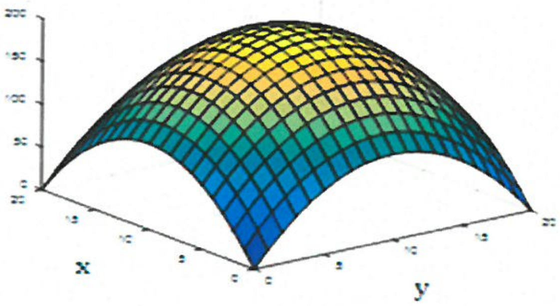
$\rightarrow \max = \underline{\underline{200}}$

First-best sol^n

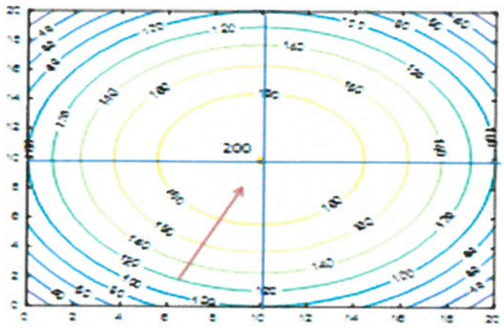
$\rightarrow y = 10 - x$

Restriction of Domain set of (x,y) that you can choose.

3-D curve of the objective function



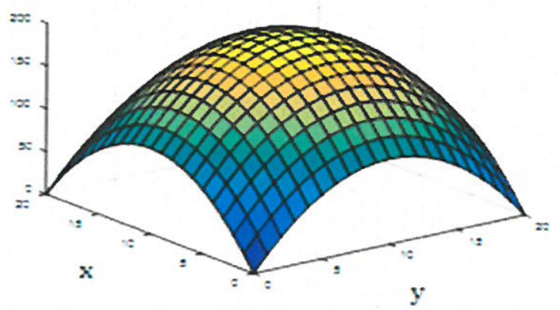
2D representation of the objective function



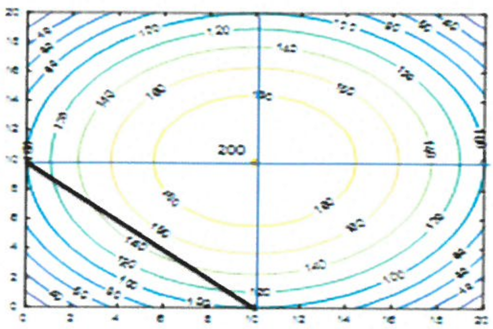
X= 10 and y = 10 is the optimum point under constraint-free optimization problem.

Introducing a constraint set would affect the possible choice set.

3-D curve of the objective function

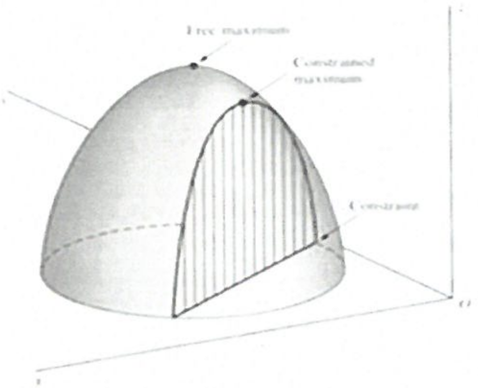


2D representation of the objective function



Can we still choose x = 10 and y = 10 when the constraint is imposed?

The Impact of introducing constraint



- Lower the optimum value of the objective function.

- Our goal is to come up with solution methods that help finding the constrained optimum point.

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① Substitution method

$\max \ f(x,y)$

s.t. $g(x,y) = c \rightarrow y = h(x,c)$

$\frac{\ln(x+y)}{e^{x+y}} \cdot \frac{x}{y} + 4 = 4xy$

$f(x, h(x,c))$ → lower dimension unconstrained optimization

② LaGrange Method.

Solution methods

1) Substitution method

Example:

$$\max_{x,y} \quad x \quad y$$

subject to $\quad x + y = 7$

Limitations:

- Functional form of the constraint function is too complicate.
  - Ex. $\ln\left(\frac{x}{y}\right) + x^2 - \frac{y^2}{x^2+1} = \frac{3}{xy} + 2$
- More than single constraint set.
  - In general case, there can be more than one constraint.
  - Doable, but difficult/impractical to reduce the problem into unconstrained optimization problem.

2) LaGrange method

Construct a new function called the **LaGrange function**.

max (min) (x,y) $f(x,y)$
s.t. $g(x,y)=c$

The function takes the following form:

a new variable

$$L(x, y, \lambda; c) = \underbrace{f(x, y)}_{\text{obj. } f^2} + \lambda[\underbrace{\boldsymbol{c} - \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y})}_{\text{Constraint set } f^2}]$$

λ is the LaGrange multiplier.

(an auxiliary variable introduced for computational purpose, to be explained later.)

Sol^2 to Constrained optimization is
the stationary sol^2 to the LaGrange f^2

- Under some regularity conditions, the solution to the constrained optimization is the stationary point for the LaGrange function.
  - A stationary point of a function is the point where "first-order derivatives" are equal to zero.
- For the LaGrange function, they are (x^*, y^*, λ^*) such that

(i) Constraint def holds

(ii) Tangency Condition

$$\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial \lambda} = 0$$

slope of level set

$$\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = \frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}}$$

slope of constraint

$$\mathcal{L} = f(x,y) + \lambda(c - g(x,y))$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial f}{\partial x} + \lambda\left(-\frac{\partial g}{\partial x}\right) = 0$$

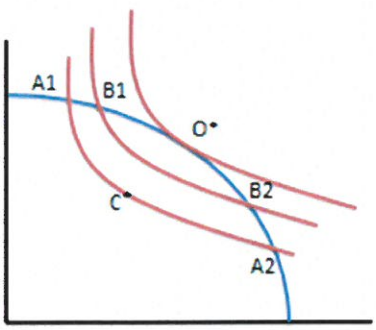
$$\frac{\partial \mathcal{L}}{\partial y} = \frac{\partial f}{\partial y} + \lambda\left(-\frac{\partial g}{\partial y}\right) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = c - g(x,y) = 0 \rightarrow$$

(x, y) must be on constraint set

Computationally, solve for (x^*, y^*, λ^*) that makes the three equations satisfied.

A geometric interpretation



- Keep moving the objective function until the function is tangent to the constraint set.

choice variable parameters

$(x,y) \rightarrow (c_0, c_1, c_2)$

Example: Solve for the problem $\max_{x,y} x \quad y \quad s.t. \quad c_0 x + c_1 y = c_2$

$$\mathcal{L} = xy + \lambda[c_2 - c_0 x - c_1 y]$$

$(x^*, y^*, \lambda^*) \rightarrow$ being sol^n to constraint opti....

3 first-order conditions.

$$\frac{\partial \mathcal{L}}{\partial x} = y + \lambda[-c_0] = 0 \quad -① \rightarrow y = \lambda \cdot c_0 \rightarrow \lambda = \frac{y}{c_0}$$

$$\frac{\partial \mathcal{L}}{\partial y} = x + \lambda[-c_1] = 0 \quad -②$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = c_2 - c_0 x - c_1 y = 0 \quad -③$$

Constraint Equation

①/② $\quad \frac{y}{x} = \frac{c_0}{c_1} \quad -④$

$$y = \frac{c_0}{c_1} \cdot x$$

→ From #3 $\quad c_2 - c_0 x - c_1\left(\frac{c_0}{c_1} \cdot x\right) = 0$

$$x^* = \frac{c_2}{2c_0}, \quad y^* = \frac{c_2}{2c_1}$$

$$\lambda^* = \frac{c_2}{2 \cdot c_1 c_0}$$

↳ Sol^n of Constraint maximization problem.

The Interpretation of LaGrange multiplier

Definition: Optimal value function

$f^*(\mathbf{c}) = f(x^*(\boldsymbol{c}), y^*(\boldsymbol{c}))$ as the optimal value of the object function.

Replace x^*, y^* back into objective f^*

we obtain maximum value of the objective f^*.

$$f^*(x,y) = x^* \cdot y^* = \left(\frac{c_2}{2c_0}\right)\left(\frac{c_2}{2c_1}\right) \Rightarrow \frac{c_2^2}{4c_0 c_1}$$

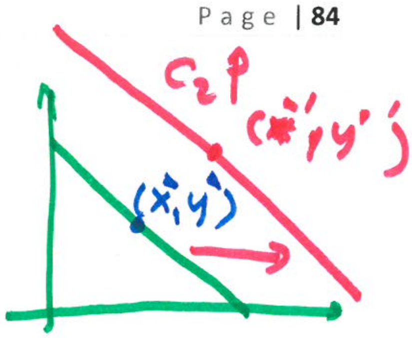
$\}$ optimal value f^*

$$\frac{\partial f^*}{\partial c_2} = \frac{2 \cdot c_2}{4 \cdot c_0 c_1}$$

$$\frac{\partial f^*}{\partial c_1} = -\frac{c_2^2}{4 c_0 c_1^2}$$

$$\frac{\partial f^*}{\partial c_0} = -\frac{c_2}{4 c_0^2 c_1}$$

doing to check the sensitivity of the optimal value f^* with respect to the parameters in the constrained optimization problem.



Envelop theorem: Given the optimal value function,

$$\frac{df^*(\boldsymbol{c})}{dc_i} = \frac{dL(x, y; \boldsymbol{c})}{dc_i}$$

Proof:

$$\mathcal{L}(x, y, \lambda, c) = xy + \lambda[c_2 - c_0 x - c_1 y]$$

$$\frac{df^*}{dc_0} = \frac{d\mathcal{L}}{dc_0}\Big|_{x^*, y^*, \lambda^*} \Rightarrow -\lambda^* \cdot x^*$$

$$\lambda^* = \frac{c_2}{2c_1c_0} \qquad x^* = \frac{c_2}{2\cdot c_0}$$

$$-\left(\frac{c_2}{2c_1c_0}\right)\left(\frac{c_2}{2\cdot c_0}\right)$$

$$\frac{df^*}{dc_0} = -\frac{c_2^2}{4c_1\cdot c_0^2}$$

shadow value of the constraint = $\lambda^* = \frac{df^*}{dc_2} \Rightarrow$ Sensitivity of f^* (optimal value f^*) when you relax the constraint set by a unit.

Following the theorem, one knows that $\boldsymbol{\lambda^*(c) = \frac{df^*(c)}{dc_2}}$

By approximation, we know that: $g(x) = f(x_0) + f'(x_0)\cdot(x - x_0)$

$\lambda^*(c) = \frac{df'(c)}{dc_2}$

$$f^*(c + dc_2) - f^*(c) \approx \underbrace{\lambda^*(c)}_{\text{per unit change}} dc_2$$ → units of change.

⇒

If we relax the constraint set/value (c_2) imposed on our optimization by a small amount of dc_2 units, on its *first approximation*, the optimized value of the objective function would be changed by $\lambda^*(c)\ dc_2$.

The second-order condition: confirming the result

- Similar to the unconstrained optimization problem, we need to confirm our result.
- If the objective function is <u>concave</u> at the stationary point (along the constraint set), solution to the first-order condition is then confirmed to be a constrained maximizer.
- To do this, we check for the property of the second-order derivative matrix of the LaGrange function.
- By checking the property of the so called "*Bordered*" Hessian matrix.

$$\bar{H} = \begin{bmatrix} L_{\lambda\lambda} & L_{\lambda x} & L_{\lambda y} \\ L_{x\lambda} & L_{xx} & L_{xy} \\ L_{y\lambda} & L_{yx} & L_{yy} \end{bmatrix}$$

→ Second-order differentiate of the LaGrange f^n

Example: Check the concavity and convexity of the example above.

$$\mathcal{L} = f(x,y) + \lambda(c - g(x,y))$$

$$\bar{H} = \begin{bmatrix} \mathcal{L}_{\lambda\lambda} & \mathcal{L}_{\lambda x} & \mathcal{L}_{\lambda y} \\ \mathcal{L}_{x\lambda} & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ \mathcal{L}_{y\lambda} & \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{bmatrix} = \cancel{\begin{bmatrix} 0 & -g_x & -g_y \\ -g_x & f_{xx} - \lambda g_{xx} & \\ -g_y & & \end{bmatrix}}$$

$$= \begin{bmatrix} 0 & -g_x & -g_y \\ -g_x & f_{xx} - \lambda g_{xx} & f_{xy} - \lambda g_{xy} \\ -g_y & f_{xy} - \lambda g_{xy} & f_{yy} - \lambda g_{yy} \end{bmatrix}$$

Checking conditions:

- For maximum, determinant of the bordered Hessian must be <u>POSITIVE.</u>
- For minimum, determinant of the bordered Hessian must be <u>NEGATIVE.</u>

Some economics examples:

Example: Regulated v.s. Unregulated monopolist

Suppose that a monopolist profit function is given by

$$\pi(x, y) = 64x - 2x^2 + 4xy - 4y^2 + 32y - 14$$

where "x" is the level of output sold to the first type of consumer and "y" is the level of output sold to the second type of consumer.

a) Find the optimal level of output that generate the highest profit

$(x^*, y^*) \rightarrow$ Constraint-free Sol^n $\rightarrow$ max π

$\boxed{\text{1-st condition:}}$

$$\frac{\partial \pi}{\partial x} = 64 - 4x + 4y = 0 \quad -①$$

$$\frac{\partial \pi}{\partial y} = 4x - 8y + 32 = 0 \quad -②$$

$① + ② \Rightarrow 96 - 4y = 0 \rightarrow y^* = 24$ units

$x \Rightarrow 4x - 8(24) + 32 = 0$

$4x = 160 \rightarrow x^* = 40$ units

Total output $\Rightarrow 40 + 24 = 64$ units.

$\boxed{\text{2-nd Condition}}$:

$$H = \begin{bmatrix} \pi_{xx} & \pi_{xy} \\ \pi_{yx} & \pi_{yy} \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ 4 & -8 \end{bmatrix}$$

$|H_1| = -4 < 0$; $|H_2| = 16 > 0 \rightarrow H$: negative def.

$d^2\pi < 0$

$\hookrightarrow \pi$ is Concave.

$\hookrightarrow (40, 24) \rightarrow$ max

$\pi^*(40, 24) =$ ←

$$\boxed{\text{Bordered Hessian Matrix}} = \begin{bmatrix} L_{\lambda\lambda} & L_{\lambda x} & L_{\lambda y} \\ L_{x\lambda} & L_{xx} & L_{xy} \\ L_{y\lambda} & L_{yx} & L_{yy} \end{bmatrix}$$

$$= \begin{array}{ccccc} & & +4 & 0 & +8 \\ \left[\begin{matrix} 0 \\ -1 \\ -1 \end{matrix}\right. & \begin{matrix} -1 \\ -4 \\ 4 \end{matrix} & \left.\begin{matrix} -1 \\ 4 \\ -8 \end{matrix}\right] & \begin{matrix} 0 \\ -1 \\ -1 \end{matrix} & \begin{matrix} -1 \\ -4 \\ 4 \end{matrix} \\ & & 0 & +4 & +4 \end{array}$$

$$\Rightarrow \bar{H} = 4 + 0 + 8 + 0 + 4 + 4 = 20$$

→ Sign is greater than zero

→ matches with the sufficient condition for the maximum sol^n.

b) What happen if the government limits the total of quantity output to be equal to 79 units.

$\hookrightarrow x+y=79 \rightarrow$ Regulatory Constraint imposed!

$$\max_{(x,y)} \pi(x,y)$$
$$\text{s.t. } x+y=79$$

LaGrange Method.

$$\mathcal{L}(x,y,\lambda) = 64x - 2x^2 + 4xy - 4y^2 + 32y - 14 + \lambda[79 - x - y]$$

(x,y,λ) $\quad \mathcal{L}_x = \mathcal{L}_y = \mathcal{L}_\lambda = 0$

$\mathcal{L}_x = 64 - 4x + 4y - \lambda = 0 \quad$ —①

$\mathcal{L}_y = 4x - 8y + 32 - \lambda = 0 \quad$ —②

$\mathcal{L}_\lambda = 79 - x - y = 0 \quad$ —③ $\rightarrow y = 79 - x$

①-②; $\quad 32 - 8x + 12y = 0 \quad$ —④

$32 - 8 \cdot x + 12(79 - x) = 0$

$32 + 12 \cdot 79 = 20x \rightarrow x^* = 49$ units.

$\lambda^* = -12 \qquad y^* = 30$ units

c) If the government allows for the production limit to be 80 units, what would happen to the optimized level of profit?

$\pi^*(79) \Rightarrow$ one thing $\boxed{}$

$\pi^*(80) \downarrow$

$x + y = 79$

$c_0 = 1$

$c_1 = 1$

$c_2 = 79$

$$\frac{d\pi^*}{dc_2} = \lambda^*$$

$$= -12$$

$\therefore c_2 \uparrow 79 \rightarrow 80$ units drop by.

$\pi^*(79) \longrightarrow \pi^*(80)$ by $12^\$$

Example: **Utility maximization problem** Budget : $P_x \cdot X + P_y \cdot y = M$

Suppose that a household has the preference relationship defined by the utility function $U(x, y) = 2x^{0.6}y^{0.3}$. Suppose further that the price of goods x is P_x and the price of goods y is P_y. Given M as the budget of this household, find the optimal bundle of consumption for goods x and goods y.

$$\mathcal{L} = 2 \cdot x^{0.6} y^{0.3} + \lambda \left[M - P_x \cdot x - P_y y \right]$$

1-st

$$\mathcal{L}_x = \mathcal{L}_y = \mathcal{L}_\lambda = 0 \Rightarrow MU_x$$

$$\mathcal{L}_x = \underline{2 \cdot (0.6) x^{-0.4} \cdot y^{0.3}} - \lambda \cdot P_x = 0$$

$= MU_y$

$$\mathcal{L}_y = 2(0.3) x^{0.6} y^{-0.7} - \lambda \cdot P_y = 0$$

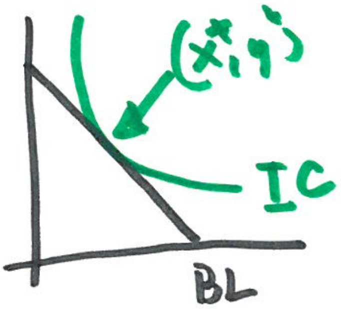
$$\mathcal{L}_\lambda = M - P_x \cdot x - P_y \cdot y = 0$$

$$\left.\frac{MU_x}{MU_y} = \frac{P_x}{P_y}\right\}$$

MRS = Price ratio.

Tangency condition holds

IC tangent to BL



Slope IC = slope BL