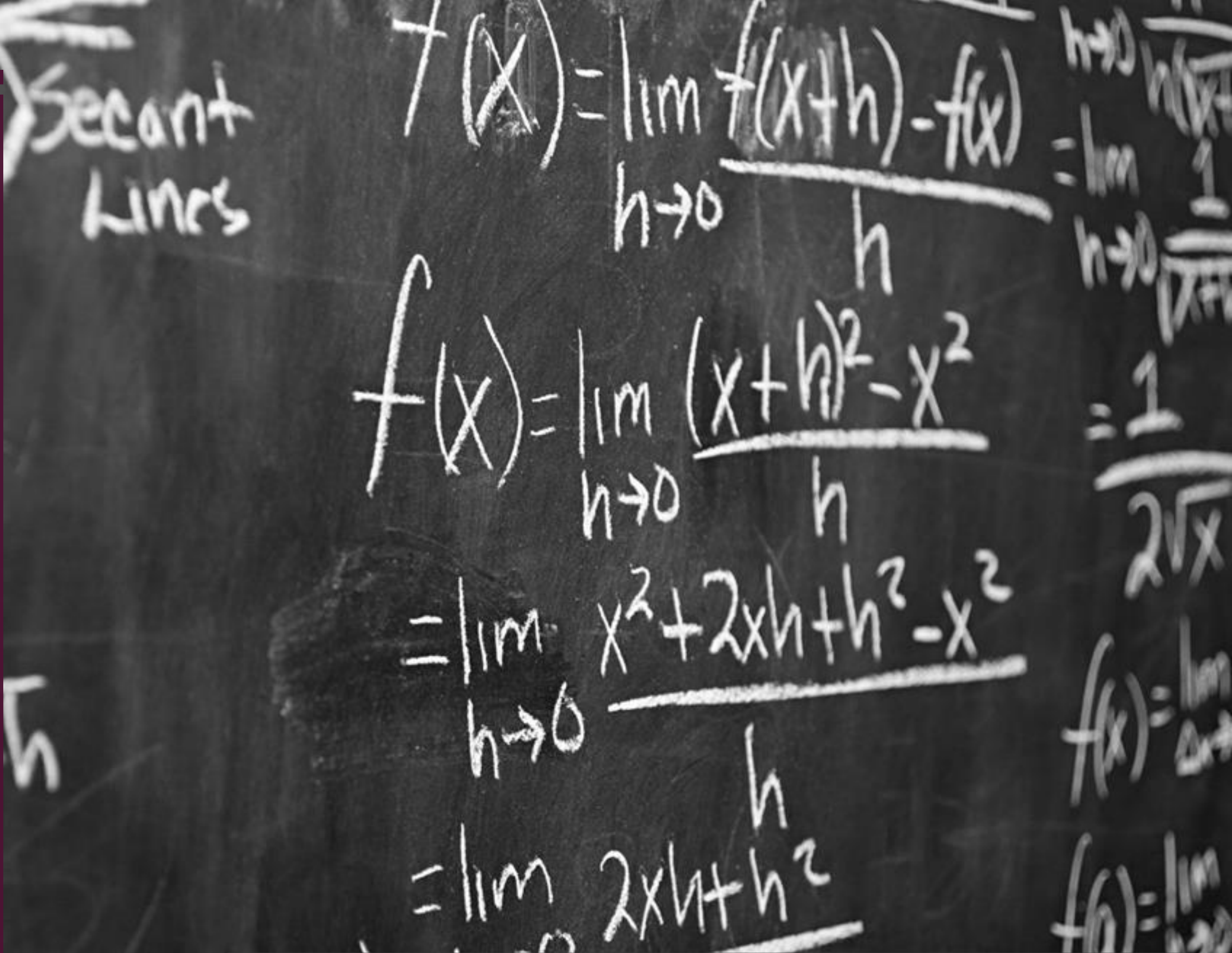
REFORMULATING PROSPECT THEORY TO BECOME A VON NEUMANN–MORGENSTERN THEORY

JACK CLARK FRANCIS

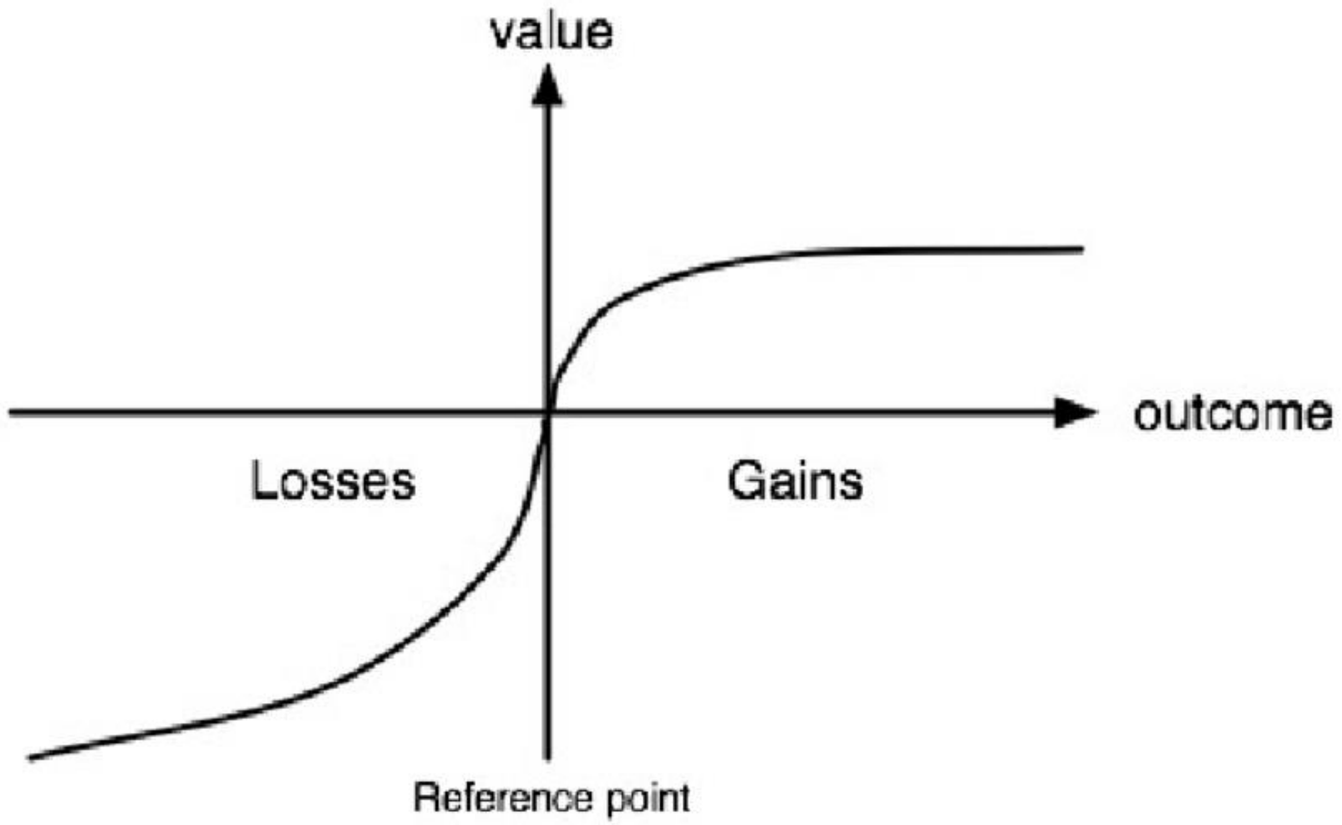


Chalisa / Chanamon / Jittrapat / Leonardo

THE PROBLEM WITH PROSPECT THEORY: KT

- Probability Weighting makes it possible for the probabilities to not add up to 1: Subcertainty.

- This means the Kahneman-Tversky value function (KT) is not a Von Neumann–Morgenstern utility function (vNM).



THE SOLUTION: THE CUBE ROOT VNM

BUT WHY?

WHY DID WE CHOOSE FRANCIS' PAPER?

Knowledge: Deepen our knowledge of Prospect Theory
and Behavioral Finance

Curiosity: How can the cube root function ameliorate Prospect Theory?

WHAT IS A VNM (VON NEUMANN–MORGENSTERN) UTILITY FUNCTION?

NOT IMPORTANT, JUST A SIMPLE REVIEW.

WHAT IS A VNM (VON NEUMANN–MORGENSTERN) UTILITY FUNCTION?

- It is a Utility function that is:
  - Complete
  - Transitive
  - Independent
  - Continuous

- If this is true, then we can use the expected utility theory on it.

- For example, $x^{1/3}$

WHY USE A VNM FUNCTION INSTEAD OF KT?

Mathematical Models

Standard deviation

Variance-covariance matrix

Classical probability theory

Etc.

Expand and accelerate research and real-world applications in

Economics

Finance

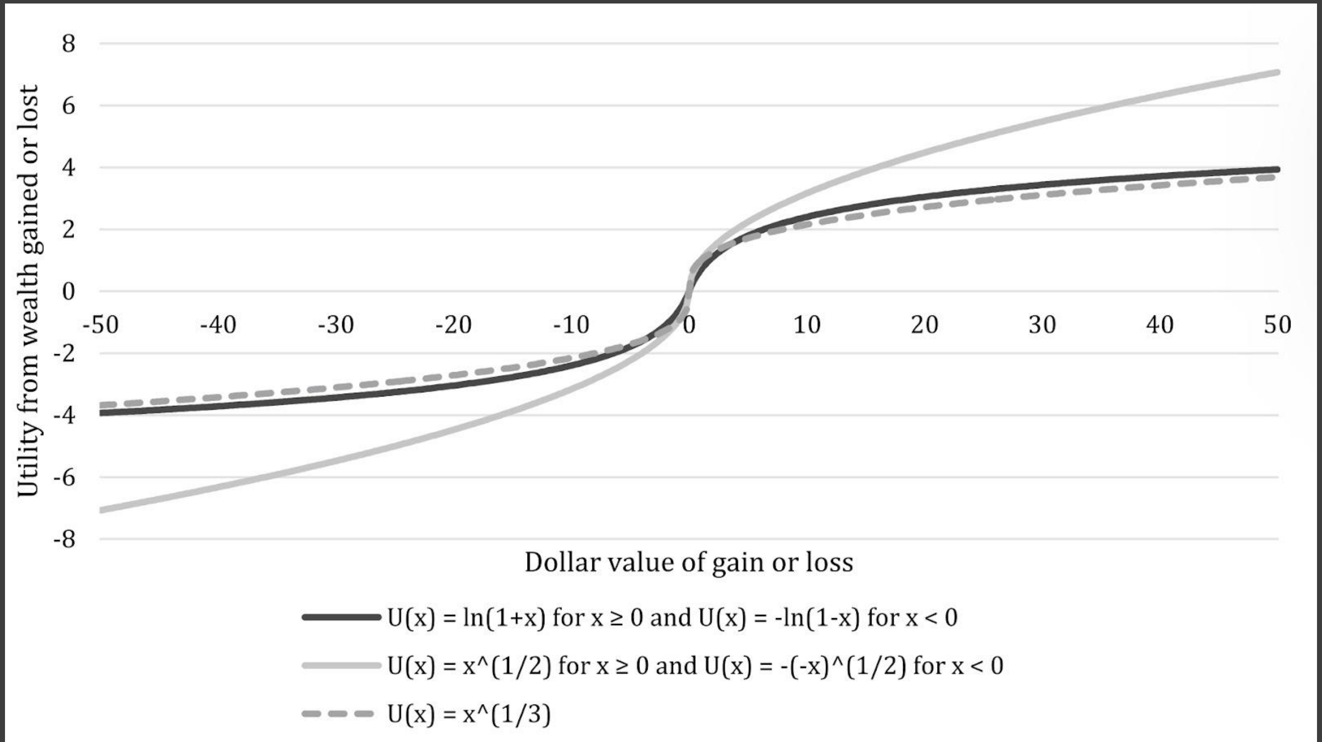
Psychology

Etc.

SHOULD WE ABANDON KT FUNCTIONS THEN?

NO, BUT WE SHOULD BRANCH THE FIELD.

QUALIFIED AND UNQUALIFIED UTILITY FUNCTIONS



UNQUALIFIED

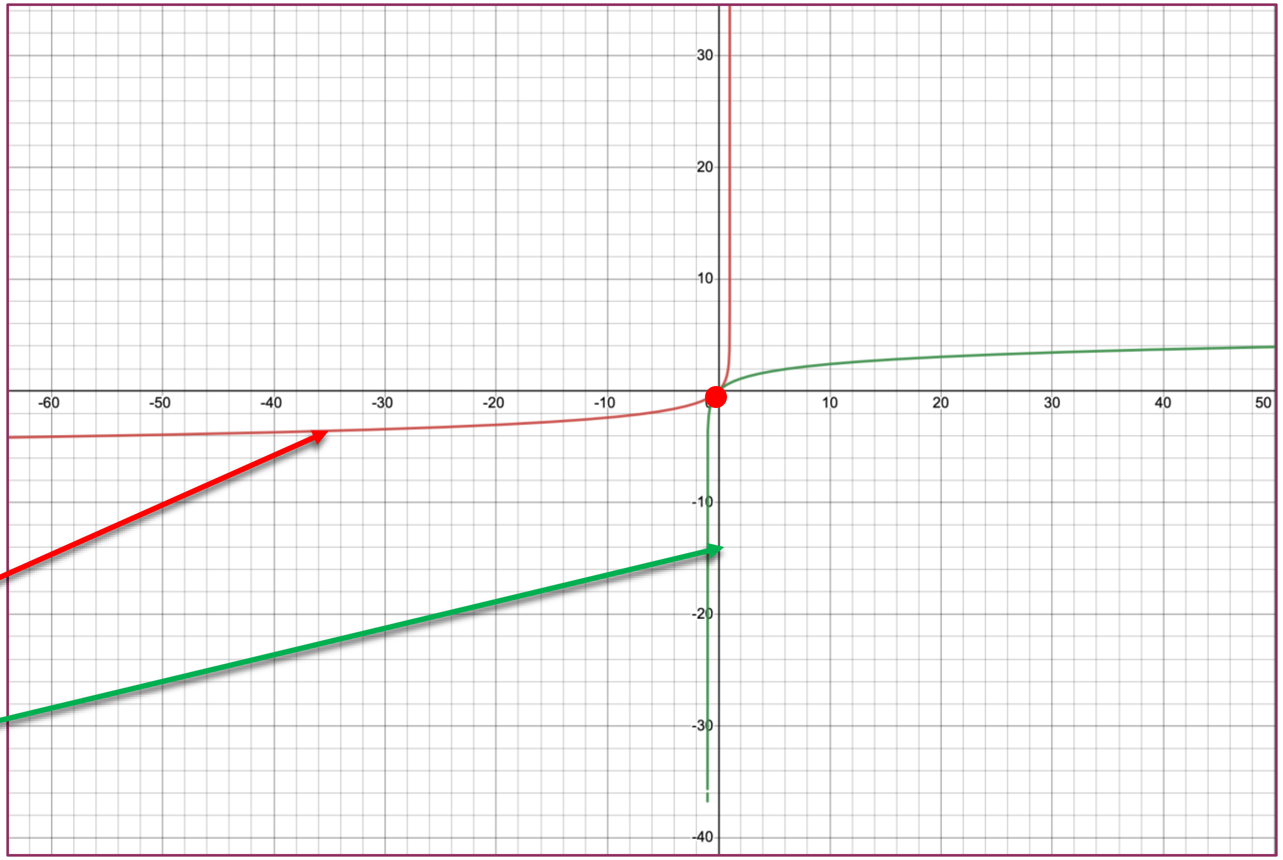
- Log function
  - Undefined over losses

$\ln(-3) = \text{Error}$

  - Create artificial S-shaped curve

$U(x) = -\ln(1-x), \quad x < 0$

$U(x) = \ln(1+x), \quad x \geq 0$



UNQUALIFIED

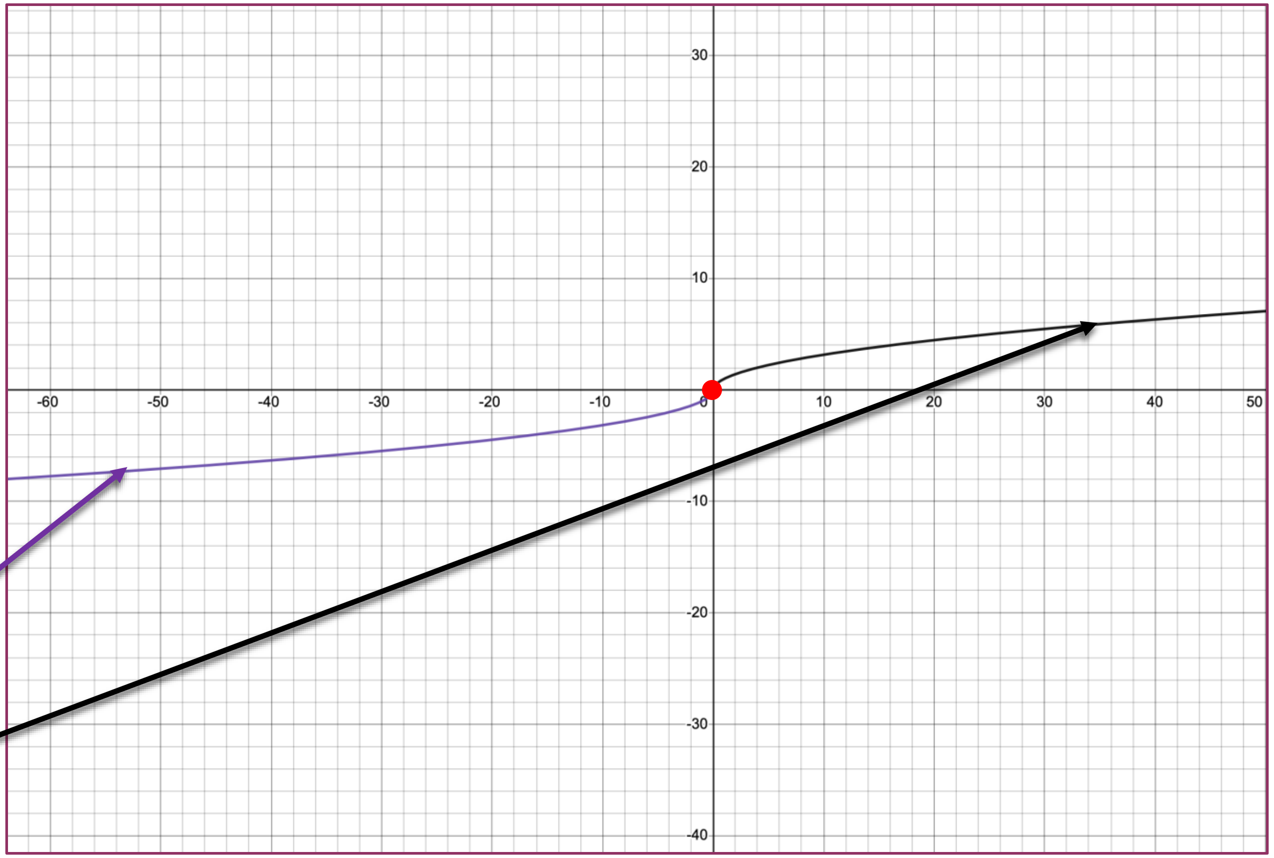
- Square root function
  - Undefined over losses

$-(-3)^{1/2}$ or $\sqrt{(-3)}$ = Error

  - Create artificial S-shaped curve

$U(x) = -(-x)^{1/2}, \quad x < 0$

$U(x) = x^{1/2}, \quad x \geq 0$



QUALIFIED

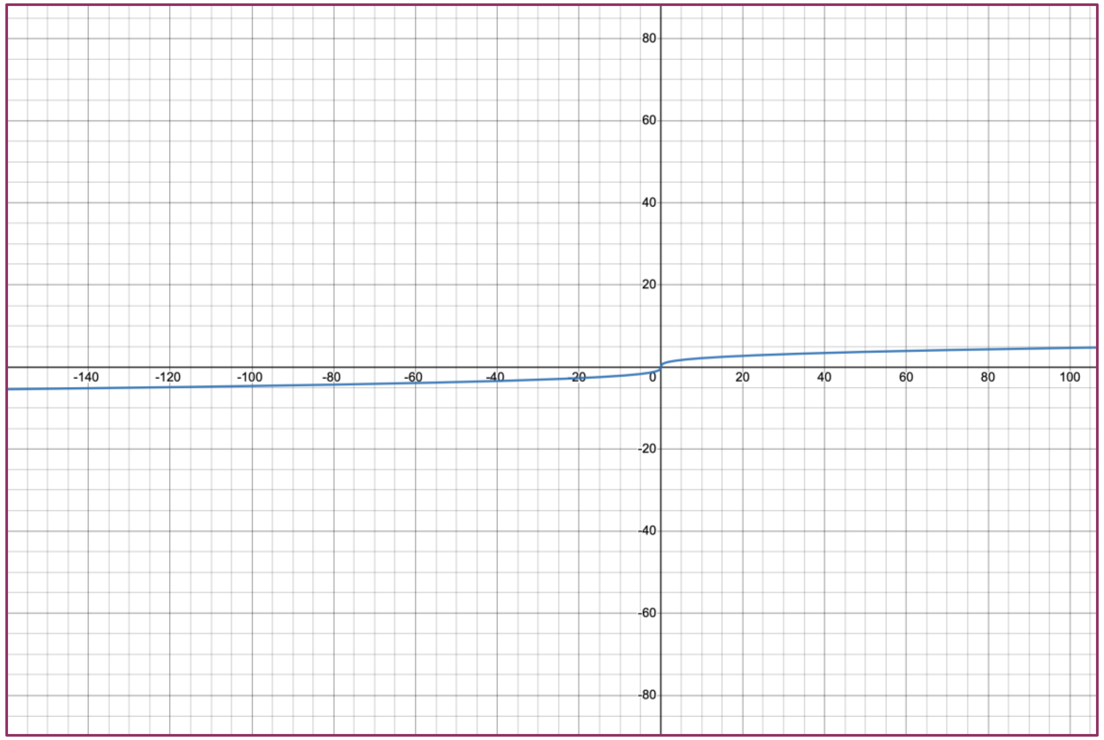
- Radical functions : $y = x^{1/n}$ or $y = \sqrt{x}$
  - 'n' must be positive odd integers
- Cube root function
  - Can analyze losses

    e.g., $U(-3) = (-3)^{1/n}$

    If $n = 4$; $U(-3) = (-3)^{1/4}$ = Error

    If $n = 5$; $U(-3) = (-3)^{1/5} = -1.2457$
  - Spans over losses and gains between

    $-\infty$ to ∞ in a single line



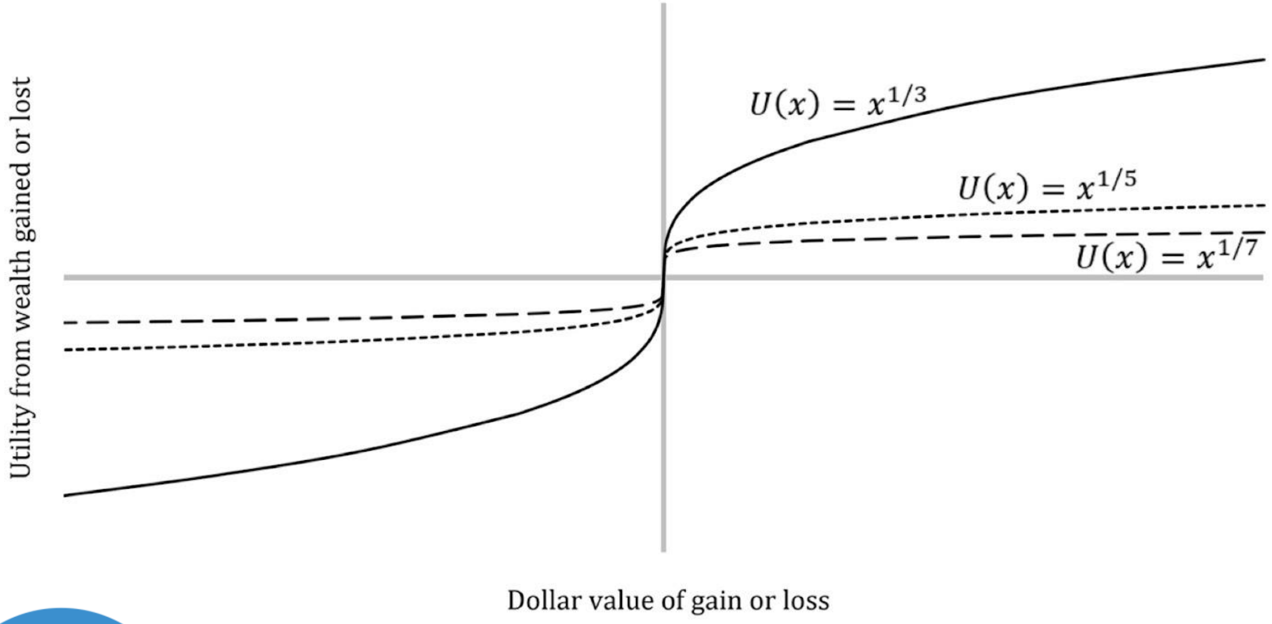
The cube root utility function: desirable outcomes and further applications

THE CUBE ROOT FUNCTION

Why is the cube root function is useful?

THE SELECTED UTILITY FUNCTION

Figure 3



Three qualified S-shape utility functions

All functions generate similar economic behavior

The cube root utility function can span wide range of utility outcomes which represent the result more clearly and more informative.

ANALYSIS OF THE CUBE ROOT

$$U(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$$

Provides a single continuous between positive and negative infinity that is kinked at the origin function.

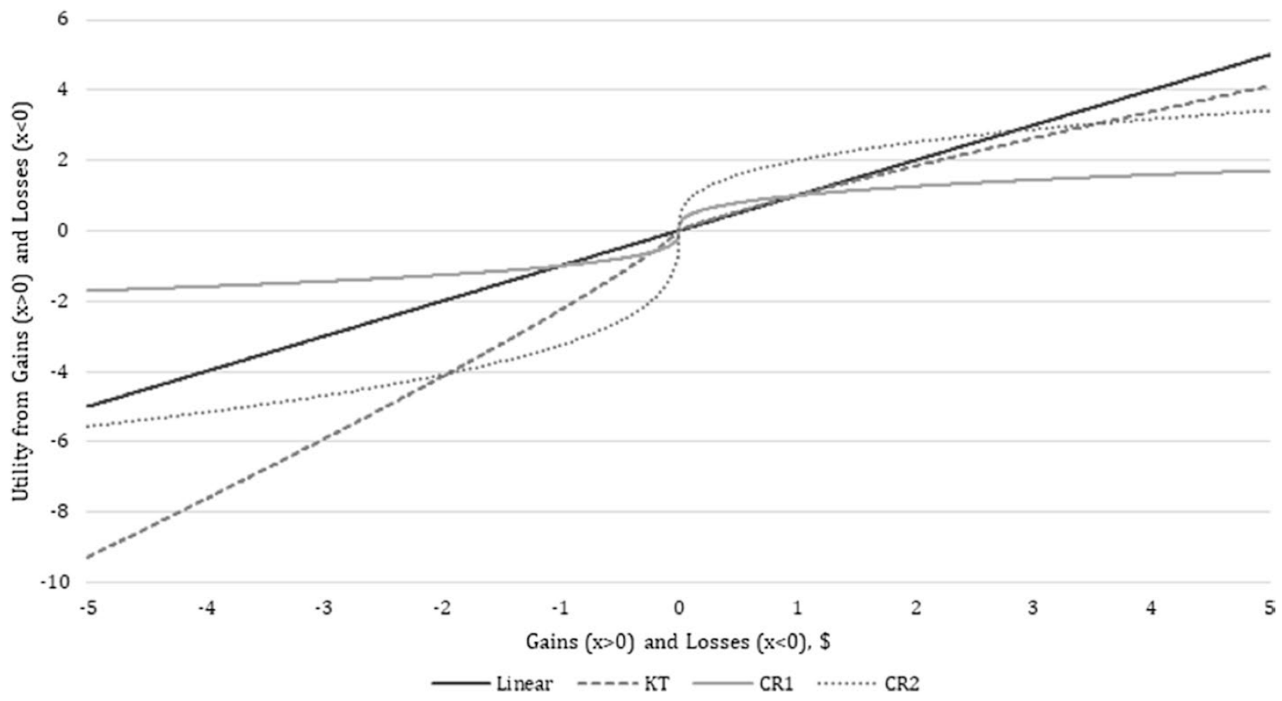
Can be formulated to be a well behaved vNM utility function.

- When **x>0**, Investors become **risk averse** in positive framework.
- When **x<0**, Investors become **risk-seeking** in negative framework.

When people asked to choosed positive and negative environments that have outcome with the same absolute value it is refers as **Cognitive biases**.

ANALYSIS OF THE CUBE ROOT

Figure 1



Implications :

1. The concave and concex segments linked at the origin to form a single function.

2. Concave = Risk averse, Happy frame
Convex = Risk-seeking, Unhappy frame

3. The kinked at the origin in a utility function creates loss aversion.

CR1 : $\gamma 1 = \gamma 2 = 1/3$ and $\alpha 1 = \alpha 2 = 1$. KT: $\gamma 1 = \gamma 2 = 0.88$ and $\alpha 2 = 2.25$. CR2 : $\gamma 1 = \gamma 2 = 1/3$ and $\alpha 1 = 2$ and $\alpha 2 = 3.25$

ANALYSIS OF UTILITY FUNCTION

UTILITY FUNCTION WITH A FAVORABLE OUTCOME, X > 0

$$U(x) = \sqrt[3]{x} = x^{\frac{1}{3}} \quad (1)$$

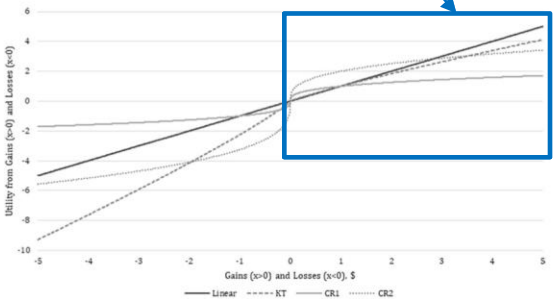
Our focus

$$= \begin{cases} \alpha_1 x^{\gamma_1}, x \geq 0, \alpha_1 = 1, \gamma_1 = 1/3 \\ \alpha_2 x^{\gamma_2}, x < 0, \alpha_2 = 1, \gamma_2 = 1/3 \end{cases} \quad (1a)$$

$$= \begin{cases} x^{1/3} \\ -(-x)^{1/3} \end{cases} \quad (2)$$

We will focus on the positive segment of S-shaped...

$$U(x) = \alpha_1 x^{1/3}, \quad x \geq 0, \quad \alpha_1 > 0$$



When x > 0
The Marginal Utility is positive

The positive sign in the first derivative

$$U'(x) = \frac{d}{dx}\left[\alpha_1 x^{1/3}\right] = \frac{\alpha_1}{3x^{2/3}} > 0, \quad \alpha_1 > 0, \quad x > 0$$

*The sign is positive

- determines a person's utility under the favorable situation
- Prefers large gains over small gains

The negative sign in the second derivative

$$U''(x) = \frac{d^2}{dx^2}\left[\alpha_1 x^{1/3}\right] = -\frac{2\alpha_1}{9x^{5/3}} < 0, \quad \alpha_1 > 0, \quad x > 0$$

*The sign is negative

- The Diminishing Marginal Utility
- Wealth Maximizing risk aversion behavior

The third derivative

$$U'''(x) = \frac{d^3}{dx^3}\left[\alpha_1 x^{1/3}\right] = \frac{10\alpha_1}{27x^{8/3}} > 0, \quad \alpha_1 > 0, \quad x > 0$$

*The sign is positive

THE PATT-ARROW MEASURE OF ABSOLUTE RISK-AVERSION (ARA), X > 0

Determine how an investor's preference for risks varies in response to changes in their wealth

Inferior to decreasing ARA

1.Decreasing ARA (DARA)

2.Constant ARA (CARA)

3. Increasing ARA (IARA)

$$ARA(x) = -\frac{U''}{U'} = \frac{2}{3x} > 0, \quad \alpha_1 > 0, \quad x > 0$$

Cube root utility function

$$ARA'(x) = \frac{d}{dx}[ARA(x)] = \frac{d}{dx}\left[\frac{2}{3x}\right] = -\frac{2}{3x^2} < 0, \quad \alpha_1 > 0, \quad x > 0$$

DARA over the range of positive outcome

THE PATT-ARROW MEASURE OF RELATIVE RISK-AVERSION (RRA), X > 0

Determine the degree to which an investor's preference for risk changes

Superior than other

1.Decreasing RRA (DRRA)

2.Constant RRA (CRRA)

3. Increasing RRA (IRRA)

$$RRA(x) = xARA(x) = \frac{2}{3} > 0$$

Cube root utility function

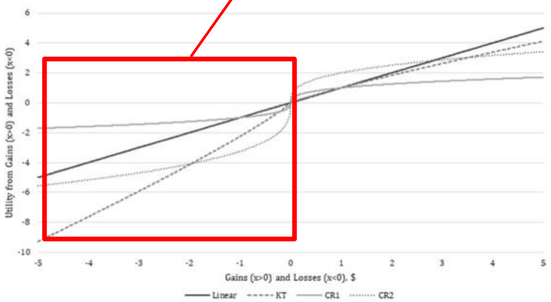
$$RRA'(x) = \frac{d}{dx}[RRA(x)] = \frac{d}{dx}\left[\frac{2}{3}\right] = 0$$

CRRA over the range of positive outcome

UTILITY FUNCTION WITH A UNFAVORABLE OUTCOMES, X < 0

We will focus on the loss segment

$$U(x) = \alpha_2 x^{1/3}, \quad x\langle 0, \alpha_2 \rangle 0$$



The positive sign in the first derivative

$$U'(x) = \frac{d}{dx}\left[\alpha_2 x^{1/3}\right] = \frac{\alpha_2}{3x^{2/3}} > 0, \quad x < 0, \quad \alpha_2 > 0$$

*The sign is positive

- determines a person's utility under the unfavorable situation
- Prefers small loss than larger loss

The negative sign in the second derivative

$$U''(x) = \frac{d^2}{dx^2}\left[\alpha_2 x^{1/3}\right] = -\frac{2\alpha_2}{9x^{5/3}} > 0, \quad x < 0, \quad \alpha_2 > 0$$

*The sign is positive

- The line is convex
- When losses occur, risk-seeking behavior frequently takes over

The third derivative

$$U'''(x) = \frac{d^3}{dx^3}\left[\alpha_2 x^{1/3}\right] = \frac{10\alpha_2}{27x^{8/3}} > 0, \quad x < 0, \quad \alpha_2 > 0$$

*The sign is positive

THE PATT-ARROW MEASURE OF ABSOLUTE RISK-AVERSION (ARA), X< 0

Determine how an investor's preference for risks varies in response to changes in their wealth

$$ARA(x) = -\frac{U''}{U'} = \frac{2}{3x} < 0, \quad x < 0, \quad \alpha_2 > 0$$

Cube root utility function under unfavorable outcome

$$ARA'(x) = \frac{d}{dx}[ARA(x)] = \frac{d}{dx}\left[\frac{2}{3x}\right] = -\frac{2}{3x^2} < 0$$

When x<0, it experience DARA

THE PATT-ARROW MEASURE OF RELATIVE RISK-AVERSION (RRA), X < 0

Determine the degree to which an investor's preference for risk changes

$$RRA(x) = xARA(x) = \frac{2}{3}$$

RRR for the cube root utility function

$$RRA'(x) = \frac{d}{dx}[RRA(x)] = \frac{d}{dx}\left[\frac{2}{3}\right] = 0$$

Generate wealth maximization behavior

A TAYLOR SERIES EXPANSION OF $\alpha_1 X^{1/3}$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x - x_0)^2}{2!} + f'''(x_0)\frac{(x - x_0)^3}{3!} + R$$

TAYLOR SERIES EXPANSION

- Variation in utility from gains and losses
- Consistent preference of positively skewed returns

- Set expected value (x_0) to be E(x):
- Reformulate into cube root utility function then simplify:

$$f(x) \cong \alpha_1 E(x)^{\frac{1}{3}} + \frac{1}{3}\alpha_1 E(x)^{-\frac{2}{3}} \cdot (x - E(x)) + \frac{1}{3}\left(\frac{-2}{3}\right)\alpha_1 E(x)^{-\frac{5}{3}} \cdot \frac{(x - E(x))^2}{2!} + \frac{1}{3}\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)\alpha_1 E(x)^{-\frac{8}{3}} \cdot (x - E(x))^3/3!$$

$$E\left[\alpha_1 (x)^{\frac{1}{3}}\right] \cong \alpha_1 E(x)^{\frac{1}{3}} + \alpha_1 \frac{1}{3}\left(\frac{-2}{3}\right)E(x)^{-5/3} \cdot E((x - E(x))^2)/2! + \frac{1}{3}\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)\alpha_1 E(x)^{-\frac{8}{3}} \cdot E((x - E(x))^3)/3!$$

$$\alpha_1 E(x^{1/3}) \cong \alpha_1 E(x)^{\frac{1}{3}} - \alpha_1 \frac{2}{9} E(x)^{-5/3} \cdot \sigma^2/2! + \frac{10}{27}\alpha_1 E(x)^{-\frac{8}{3}} E\left((x - E(x))^3\right)/3!$$

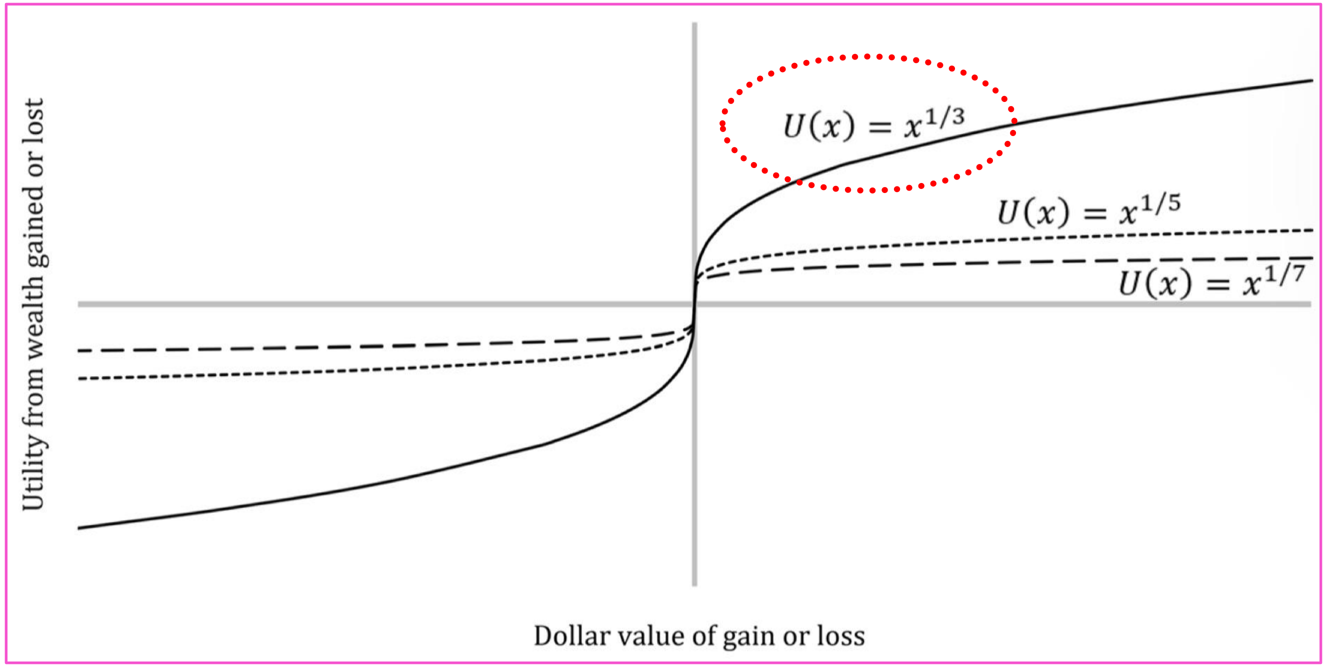
CONCLUSIONS

- *Loss aversion – Prospect theory*
  - Absolute value of utility facing losses exceeds absolute value of receiving gains in the same magnitude
  - Loss-aversion theory

- *Qualified radical function – Cube root utility function*
  - Loss aversion theories are valid
  - Nobel prize winning and social sciences theories

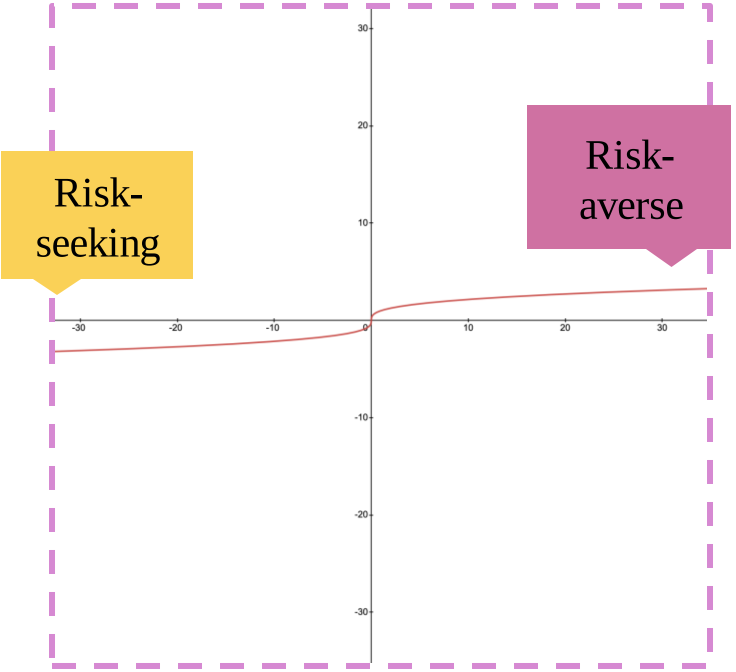
THE VNM MODEL & CUBE ROOT UTILITY THEORY

1. Single equation; convex and concave
2. Radical functions with odd integers exponents
3. Broader possible results
4. Wider applications of classical statistical methods such as standard deviation (the vNM models: probabilities sum to one)
5. Capable of interpreting more economic theories than other competing utility functions



ECONOMIC ASPECTS

1. Positive marginal utility of wealth and returns
2. Consistent risk-aversion in gain domain (concavity)
3. Consistent risk-seeking in loss domain (convexity)
4. Consistent preference of positively skewed outcome
5. Asymmetric cube root utility rationalizes loss-aversion
6. Cover all issues in KT's prospect theory except weighted probabilities



COMPARISON BETWEEN COMPETING UTILITY FUNCTIONS

- ✓ Does x span the real line between -∞ to +∞ ?
- ✓ A positive monotone function, $dU(x)/dx > 0$?
- ✓ A consistent preference for positive skewness?
- ✓ Can losses, $x < 0$, be analyzed?
- ✓ Absolute risk aversion (ARA)
- ✓ Relative risk aversion (RRA)
- ✓ Always represents rational economic behavior?
- ✓ Can this vNM function explain the complicated behaviors of KT's prospect theory?
- ➢ Total desirable yes answers **8 yes answers**

Table 1 Summary of advantages and deficiencies of competing utility functions

Criteria for a useful utility function	Five competing utility functions				
	Cube root utility function	Quadratic utility function	Square root utility function	Log utility function	Exponential utility function
Does x span the real line between $-\infty$ to $+\infty$?	Yes, with a single equation	No, the ellipse has finite limits	No, it is undefined for losses	No, it is undefined for losses	No, it is undefined for losses
A positive monotone function, $\frac{dU(x)}{dx} > 0$?	Yes, more wealth is always preferable	Yes, for one quadrant of the ellipse	No, it is undefined for losses	No, it is undefined for losses	No, it is undefined for losses
A consistent preference for positive skewness?	Yes	No skewness preference	No skewness preference	No skewness preference	No skewness preference
Can losses, $x < 0$, be analyzed?	Yes	Yes	Not able to analyze losses	Not able to analyze losses	Not able to analyze losses
Absolute risk aversion (ARA	DARA is desired	IARA is not desirable	DARA is desired	DARA is desired	CARA is not desirable
Relative risk aversion (RRA)	CRRA is desired	CRRA is desired	CRRA is desired	CRRA is desired	IRRA is not desirable
Always represents rational economic behavior?	Yes, it is never irrational	Not all segments behave rationally	No, undefined for losses	No, undefined for losses	No, CARA always exists
Can this vNM function explain the complicated behaviors of KT's prospect theory?	Yes, except the weighted probabilities	No, it has shortcomings	No, it has shortcomings	No, it has shortcomings	No, it has shortcomings
Total desirable yes answers	8 yes answers	3 yes answers	2 yes answers	2 yes answers	Zero yes answers