

EE320 (1/2013)

INTRODUCTORY MATHEMATICAL ECONOMICS

INTEGRATION AND ITS APPLICATION

Topics

- Terminology in Integration
- Indefinite Integration
- Basic Rules of Integration
- Definite Integration
- Improper Integrals

What is Integration?

- *Integration* is the inverse of differentiation.
- Formally, an integral is a function $F(x)$ whose derivative is $f(x)$:
$$F'(x) = f(x)$$
- This function $F(x)$ can then be called an 'anti-derivative' of $f(x)$.

Example:

- $f(x) = nx^{n-1} \rightarrow F(x) = x^n$
- $f(x) = \frac{1}{x} \rightarrow F(x) = \ln(x)$
- The process of anti-differentiation is called integration.
- That is, to integrate a function $f(x)$ is to find $F(x)$ such that

$$F'(x) = f(x)$$

Indefinite Integrals

- If we integrate a function $f(x)$ where values of x are not given, we have to *integrate without a limit* (i.e. to find *indefinite integral*).
- A symbol for integrating a function $f(x)$ is:

$$\int f(x)dx = F(x) + C \Leftrightarrow F'(x) = f(x)$$

Where $\int$ is called the *integral sign*.

$f(x)$ is called the *integrand*.

C is called the *constant of integration*.

dx indicates the variable involved in the integration.

- Note: A function does not have a unique integral.

Example: If $f(x) = 3x^2$, then $F(x) = x^3+1$ or x^3+7 or $x^3+ C$

Basic Rules of Integration

Rule I. Power rule: $\int x^n dx = \frac{1}{n+1} x^{n+1} + c, n \neq -1$

Rule II. Exponential rule: $\int e^{ax} dx = \frac{1}{a} e^{ax} + c$

Rule III. Logarithmic rule: $\int \frac{1}{x} dx = \ln|x| + c$

Rule IV: Integral of a sum

$$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$$

Rule V: Integral of a multiple

$$\int af(x) dx = a \int f(x) dx$$

Rules of Operations

Rule IIa:

$$\int a^{bx} dx = \frac{1}{b \ln a} a^{bx} + c$$

Rule IIb:

$$\int f(x) e^{f(x)} dx = e^{f(x)} + c$$

Rule IIIa:

$$\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + c, [f(x) \neq 0]$$

Examples

Find integrals of the following functions:

- $\int \frac{1}{x^3} dx = \frac{1}{2x^2} + c$
- $\int \sqrt{x} \sqrt{x} \sqrt{x} dx = \int x^{7/8} dx = \frac{8}{15} x^{15/8} + c$
- $\int (3x^4 + 5x^2 - 2) dx = \frac{3}{5} x^5 - \frac{5}{3} x^3 - 2x + c$
- $\int (e^{3x} - e^{2x} + e^x) dx = \frac{1}{3} e^{3x} - \frac{1}{2} e^{2x} + e^x + c$
- $\int \frac{(y-2)^2}{\sqrt{y}} dy = \frac{2}{5} y^{5/2} - \frac{8}{3} y^{3/2} + 8y^{1/2} + c$

Rules Involving Substitution

Rule VI: The Substitution Rule

$$\int f(u) \frac{du}{dx} dx = \int f(u) du = F(u) + c$$

Rule VII: Integration by Parts**

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

Let $u = f(x)$ and $v = g(x)$. Then,

$$\int v du = uv - \int u dv$$

Examples: Integration involving substitution

- Examples: Find integrals of

a. $\int x e^x dx$

Ans. $\int x e^x dx = e^x (x - 1) + c$

b. $\int \frac{1}{x} \ln(x) dx$

Ans. $\int \frac{1}{x} \ln(x) dx = \frac{(\ln x)^2}{2} + c$

Initial-Value Theorem

- From $\int f(x)dx = F(x) + C$

If we have an initial condition, we can determine the value of C.

Example 1: Find $F(x)$ if $F'(x) = \frac{1}{2} - 2x$ and $F(0) = \frac{1}{2}$.

Ans.
$$F(x) = \frac{1}{2}x - x^2 + \frac{1}{2}$$

Example 2: Find $F(x)$ if $F'(x) = x(1-x^2)$ and $F(1) = 5/12$.

Ans.
$$F(x) = \frac{x^2}{2} - \frac{x^4}{4} + \frac{1}{6}$$

Application 1: Derivation of TR from MR

- $TR = \int MR(Q)dQ$
- Example: Let $MR = 10 - 2Q$, what is the TR function?

$$\text{Ans. } TR(Q) = 10Q - Q^2 = (10-Q)Q$$

Application 2: Derivation of TC from MC

- $TC = \int MC(Q)dQ$
- Example: Let $MC = C'(Q) = 2e^{0.2Q}$.

If the fixed cost is $C_F = 90$, what is the TC function?

$$\text{Ans. } TC(Q) = 10e^{0.2Q} + 80$$

Application 3: Derivation of Profit Function from MR-MC

- $\pi' = MP = MR - MC. \rightarrow \pi = \int \pi'(Q)dQ$
- Example: Let $MR = 50 - 2Q$ and $MC = 10 + Q$. Find total profit when $Q = 10$. Assume that there is no fixed cost.

Ans. $\pi(Q) = 40Q - 1.5Q^2$

Application 4: Derivation of Utility Function from MU

- $U(x) = \int MU(x) dx$
- Example: Let $MU(x) = \frac{5}{3\sqrt{x}}$. Find the utility function.

Ans. $U(x) = \frac{10}{3}\sqrt{x} + c$

Application 5: Derivation of Consumption/Saving Functions from Marginal Propensity Function

- Suppose the marginal propensity to save (MPS) function is:

$$S'(Y) = 0.3 - 0.1Y^{-1/2}, \text{ and } S = 0 \text{ when } Y = \$81.$$

Find the saving and consumption functions.

Ans. $S(Y) = 0.3Y - 0.2\sqrt{Y} - 22.5$

$$C(Y) = 0.7Y + 0.2\sqrt{Y} + 22.5$$

Definite Integrals

- $\int_a^b f(x)dx$ is a “definite” integral of $f(x)$ from a to b ($a < b$), where
 - a = the lower limit of integration
 - b = the upper limit of integration



$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

where $F(x)$ = an arbitrary indefinite integral of $f(x)$.

- Example: Find integral of

$$\int_1^5 3x^2 dx = 124$$

$$\int_0^1 \alpha e^{\beta\tau} d\tau = \frac{\alpha}{\beta} (e^{\beta} - 1)$$

Area and Definite Integrals

- The area under the graph of a continuous and nonnegative function $f(x)$ over the interval $[a, b]$ is $\int_a^b f(x)dx$.
- [Graph]

Properties of Definite Integrals (I)

Property I: The interchange of the limits of integration changes of the sign of the definite integral:

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Property II: A definite integral has a value of zero when the two limits of integrations are identical:

$$\int_a^a f(x)dx = 0$$

Property III: A definite integral can be expressed as a sum of a finite number of definite sub-integrals as follows:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, (a < b < c < d)$$

Properties of Definite Integrals (II)

Property IV:

$$\int_a^b -f(x)dx = -\int_a^b f(x)dx$$

Property V:

$$\int_a^b \alpha f(x)dx = \alpha \int_a^b f(x)dx$$

Property VI:

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

Property VII: (Integration by part)

$$\int_{x=a}^{x=b} vdu = uv \Big|_{x=a}^{x=b} - \int_{x=a}^{x=b} u dv$$

Examples: Definite Integrals

Find the integrals of

- $\int_0^5 (x + x^2) dx = \frac{325}{6}$
- $\int_2^4 x^2 \left(\frac{1}{3} x^3 + 1 \right) dx = \frac{1184}{9}$
- $\int_{-2}^2 (e^x - e^{-x}) dx = \left(e^x - e^{-x} \right) \Big|_{-2}^2 = 0$
- $\int_e^6 \left(\frac{1}{x} + \frac{1}{1+x} \right) dx = \left(\ln x + \ln |1+x| \right) \Big|_e^6 = (\ln 6 + \ln 7) - (1 + \ln(e+1))$
- $\int_{-2}^3 |x+1| dx = \int_{-2}^{-1} -(x+1) dx + \int_{-1}^3 (x+1) dx = 0.5 + 8 = 8.5$

Application 1: Capital Formation and Investment functions

- Definitions:

- $K(t)$ = capital stock at time t
- dK/dt = the rate of capital formation
- $I(t)$ = the rate of net investment flow at time t

- Relationship between capital stock and net investment:

$$\frac{dK}{dt} = \dot{K} \equiv I(t) \quad \Rightarrow \quad K(t) = \int I(t)dt = \int \frac{dK}{dt}dt = \int dK$$

- Gross investment: $I_g(t) = I(t) + \delta K(t)$

- Capital formation during a time interval $[a, b]$:

$$\int_a^b I(t)dt = K(t) \Big|_a^b = K(b) - K(a)$$

Capital Formation and Investment functions (Cont'd)

- Example: Suppose the net investment flow is $I(t) = 3t^{1/2}$ and the initial capital stock at time $t = 0$ is $K(0) = 25$.

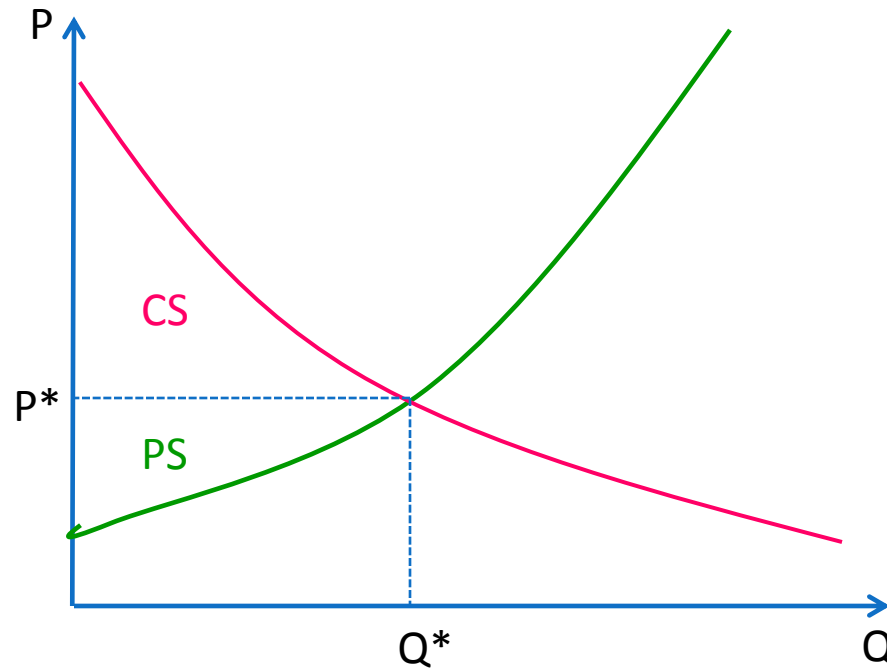
1. What is the time path of capital K ?

Ans. $K(t) = \int I(t)dt = \int 3t^{1/2}dt = 2t^{3/2} + 25$

2. What is the capital formation during the time interval $[1, 4]$?

Ans. $K(4) - K(1) = \int_1^4 I(t)dt = 14$

Application 2: Consumer & Producer Surpluses



$$CS = \int_0^{Q^*} [D(Q) - P^*] dQ$$

$$PS = \int_0^{Q^*} [P^* - S(Q)] dQ$$

Example: Consumer Surplus & Producer Surplus

- Given a supply function $S(P) = -\frac{1}{2} + \frac{1}{2}P$ and a demand function $D(P) = \frac{25}{2} - \frac{1}{2}P$. Calculate: 1) producer and consumer surplus and 2) total welfare.

➤ Ans. $P^* = 13$, $Q^* = 6$
 $CS = 36$; $PS = 36$

Welfare Effects of Price Change

- From the previous example, if the demand changes so that $D(P) = \frac{30}{2} - \frac{1}{2}P$, what happens to consumer and producer surplus?
- Ans. $P^* = 15.5$, $Q^* = 7.25$
 $CS = PS = 52.5625$

Welfare Effects of Tax Distortion

- From the previous example, if the government imposes a \$4 per unit tax on the producer, calculate the total welfare loss.

➤ Ans. $P_d^* = 15$, $P_s^* = 11$, $Q^* = 5$

$$CS = PS = 25 \rightarrow \Delta CS = \Delta PS = 25 - 36 = -11$$

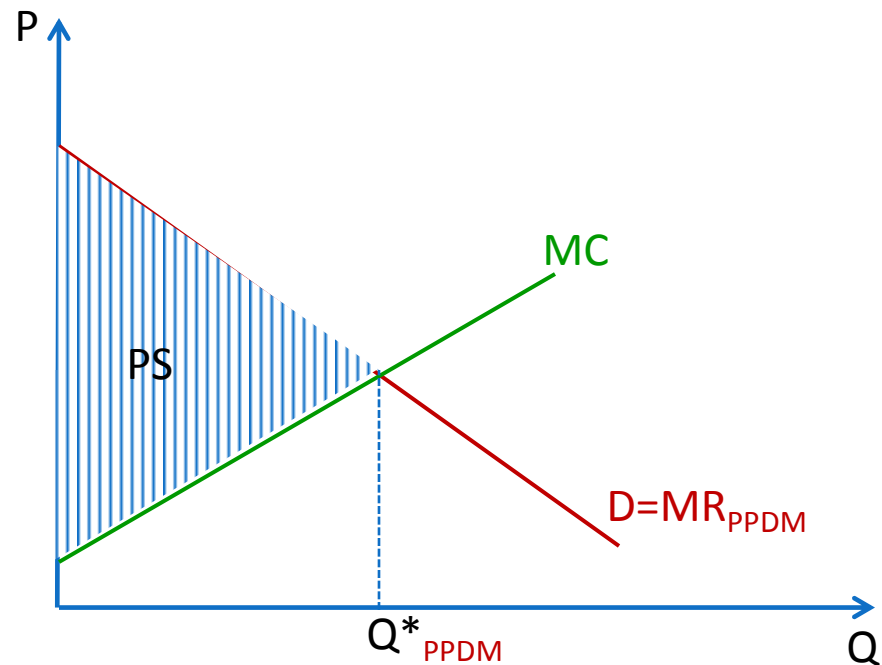
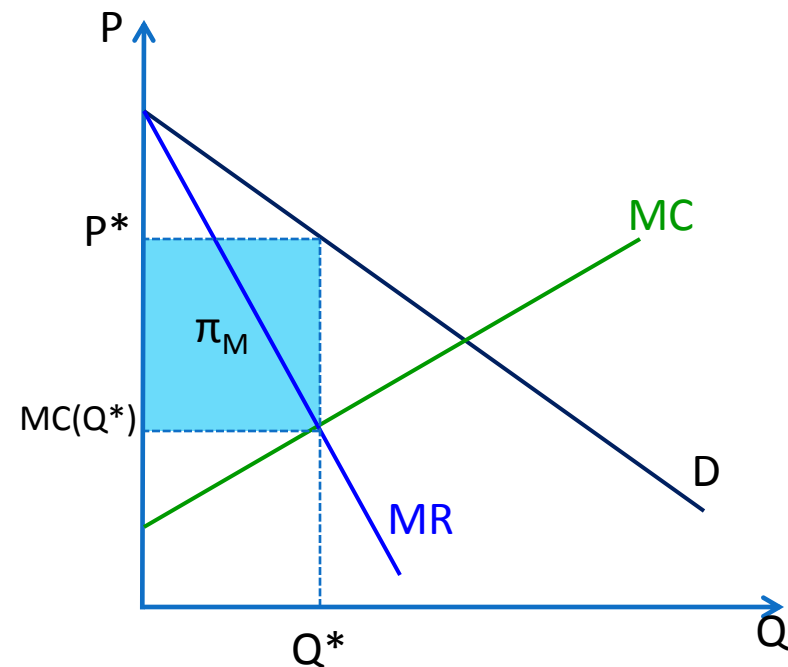
$$\text{Deadweight Loss} = \text{Total Welfare loss} - \text{Tax}$$

$$= (11 + 11) - (4 * 5)$$

$$= 2$$

Application 3: First-Degree (or Perfect) Price Discrimination

- Perfect price discrimination occurs when the monopolist can charge the *maximum price for each unit of output sold*.
- (Regular) Monopolist
- Perfect price discriminating monopolist



Example: Perfect Price Discrimination (1)

- Suppose that a monopolist faces a demand function $P = 24 - Q$ and $MC = 4 + 3Q$. Find the consumer surplus at the profit-maximizing quantity. (No price discrimination)

➤ π -max: $MR = MC \rightarrow 24 - 2Q = 4 + 3Q$

$\rightarrow Q^* = 4 ; P^* = 20 \ (\pi^* = 16)$

➤ $CS = \int_0^4 [(24 - Q) - 20] dQ = 8$

Example: Perfect Price Discrimination (2)

- If the monopolist can practice perfect price discrimination, find the total revenue that maximizes the profit.

$$\begin{aligned}\text{➤ } \pi\text{-max: } D &= MC \rightarrow 24 - Q = 4 + 3Q \\ &\rightarrow Q^* = 5\end{aligned}$$

$$\text{➤ } \pi^* = \int_0^5 [(24 - Q) - (4 + 3Q)] dQ = 75$$

$$\text{➤ } TR^* = \int_0^5 (24 - Q) dQ = 107.5$$

Improper Integrals (I)

- Case 1: When we have definite integrals of the form

$$\int_a^{\infty} f(x)dx \quad \text{and} \quad \int_{-\infty}^b f(x)dx$$

with **one limit of integration being infinite**, we refer to them as **improper integrals**. (because $F(\infty)$ and $F(-\infty)$ does not exist.)

- The improper integral can be defined to be the limit of another (proper) integral as follows:

$$\int_a^{\infty} f(x)dx \equiv \lim_{b \rightarrow \infty} \int_a^b f(x)dx$$

and

$$\int_{-\infty}^b f(x)dx \equiv \lim_{a \rightarrow -\infty} \int_a^b f(x)dx$$

- If the **limit exists**, the improper integral is said to be **convergent** and the limiting process gives the value of the integral.
- If the **limit does not exist**, the improper integral is said to be **divergent** and does not have any meaning.

Improper Integrals (II)

Examples: Evaluate the following integrals.

- $\int_1^{\infty} \frac{1}{x^2} dx$

- $\int_1^{\infty} \frac{1}{x} dx$

Improper Integrals (III)

- Case 2: Even with finite limit of integration, an integral can still be improper if the integrand becomes infinite in the interval of the integration $[a, b]$.

Example: $\int_0^1 \frac{1}{x} dx =$

$$\int_0^9 \frac{1}{x^{1/2}} dx$$

- If $f(x)$ is a continuous function for every $x \in R$, then

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^0 f(x) dx + \int_0^{\infty} f(x) dx$$

Example: $\int_{-1}^1 \frac{1}{x^3} dx =$