# Chapter 7 and 8: Relations and Functions

## 1. Order pairs and Cartesian product of A and B

**Definition 1.1**: ***Cartesian product of A and B***, denoted by $A \times B$ , is the set of all order pairs which is written as a coordinate $(x, y)$ where the first element of each order pair is an element of *A* and the second element of each order pair is an element of *B*.

**Theorem 1.1:** Let $(x, y)$ and $(a, b)$ be any order pairs. If $(x, y) = (a, b)$ then $x = a$ and $y = b$.

**Example 1:** Let $A = \{1, 2\}$ and $B = \{-1, 0, 9\}$. Find $A \times B, B \times A, A \times A,$ and $B \times B$.

## 2. Relations and Functions

**Definition 2.1**: A rule that assigns elements of B to elements of A is called a relation from A to B. The set of elements of A to which the assignments are made is called the domain of the relation. The set of elements of B used in the assignments is the range of relation.

**Definition 2.2**: A relation from set A to set B is called a mapping or a function if the relation assigns to each element of A exactly one element of B.

**Theorem 2.1**: Vertical-Line Test for a Function

An equation defines a function if each vertical line in the coordinate system passes through at most one point on the graph of the equation. If any vertical line passes through two or more points on the graph of an equation, then the equation does not define a function.

**Definition 2.3**: $f$ is said to be a function from A **into** B if $f$ is a function that has A as a domain and range of $f$ is a subset of B. A function A into B is denoted by $f: A \to B$ or $f: A \xrightarrow{\text{into}} B$.

**Definition 2.4**: $f$ is said to be a function from A **onto** B if $f$ is a function from A into B and range of $f$ is B ($R_f = B$). A function A onto B is denoted by $f: A \xrightarrow{\text{onto}} B$.

**Example 1**: Let $A = \{1, 2\}$ and $B = \{-1, 0, 9\}$. Are the following relations a function?

1) $r = \{(x, y) \in A \times B \mid x + y = x\}$
2) $t = \{(x, y) \in B \times A \mid 0 < x + y \leq 10\}$
3) $s = \{(x, y) \in B \times B \mid xy \geq 0\}$
4) $u = \{(x, y) \in A \times A \mid 3 \leq 2x + y \leq 4\}$

**Example 2**: Let $A = \{0,1,2\}$ and $B = \{0,1,4\}$. Are the following relations a function?

1) $f = \{(x, y) \in A \times B \mid y = x^2\}$

2) $g = \{(x, y) \in A \times B \mid 2x + y = 2 \vee 2x + y = 4\}$

**Example 3**: Is the following relation a function?

1) $f_1 = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y + x = 3\}$

2) $f_2 = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = \sqrt{x-1}\}$

3) $f_3 = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x + y^2 = 3\}$

**Note:** A function is a ***one-to-one function*** if and only if each second element corresponds to one and only one first element. (each $x$ and $y$ value is used only once)

**Definition 2.5**: $f$ is said to be one-to-one function from A into B, denote by $f : A \xrightarrow{1-1} B$, if $f$ is a function from A into B and $f$ is one-to-one.

**Definition 2.6**: $f$ is said to be one-to-one function from A onto B, denote by $f : A \xrightarrow[onto]{1-1} B$, if $f$ is a function from A onto B and $f$ is one-to-one.

**Example 4**: Recall Example 3. Consider the following relations and check if they are one-to-one functions.

1) $f_1 = \left\{ (x, y) \in \mathbb{R} \times \mathbb{R} \middle| y + x = 3 \right\}$

2) $f_2 = \left\{ (x, y) \in \mathbb{R} \times \mathbb{R} \middle| y = \sqrt{x - 1} \right\}$

**Definition 2.7**: The *inverse* of a function is the set of ordered pairs obtained by interchanging the first and second elements of each pair in the original function. For function $f$ , the inverse of the function is denoted by $f^{-1}$.

Note: If the original function is a one-to-one function, the inverse will be a function.

**Example 5**: Let $A=\{1,2,3,5\}$, $B=\{-2,-1,0,4\}$ and let $f$ and $g$ be functions from A to B where $f=\{(1,0),(2,-1),(3,-2),(5,0)\}$ and $g=\{(1,-2),(2,-1),(3,0),(5,4)\}$. Find $f^{-1}$ and $g^{-1}$

**Example 6**: Let $f=\{(x,y)\in\mathbb{R}\times\mathbb{R} \mid y=2x-3\}$ be a function. Find the inverse of a function $f$ and check if it is a function.

**Example 7**: Let $f$ be a function defined by $f(x)=1+\sqrt{x-2}$. Find $f^{-1}$ and check if it is a function.

**Definition 2.9: (Composition of Functions)**

If $f$ and $g$ are functions which $R_f \cap D_g \neq \varnothing$, then the composite function of $f$ and $g$, denoted by $g \circ f$, is defined by $(g \circ f)(x) = g(f(x))$ where $x \in D_f$.

**Example 12**: Let $A = \{1, 2, 3, 4, 5\}$, $B = \{-3, -2, -1, 0, 6\}$, $C = \{-4, 7, 8, 9\}$, and $f : A \rightarrow B$, $g : B \rightarrow C$ where $f = \{(1, -3), (2, -2), (3, 0), (4, -1), (5, -2)\}$ and $g = \{(-3, 7), (-2, 9), (-1, 7), (0, 7), (6, 8)\}$. Find $g \circ f$ if possible.

**Example 13**: Let $f(x)=\dfrac{2}{\sqrt{x-1}}$ and $g(x)=x^2+3$. Find $g\circ f$ and $f\circ g$ if possible.

**Example 14:** Recall Example 2.2.7. Let $f(x)=1+\sqrt{x-2}$ then $f^{-1}(x)=2+(x-1)^2$. Find $f\circ f$ and $f\circ f^{-1}$ if possible.

**Example 15**: Let $f(x) = x^2 + 5$ and $g(x) = \begin{cases} 3 - x, & x < 2 \\ \dfrac{1}{x}, & 2 \le x < 3 \end{cases}$.

Find $f \circ g$ and $g \circ f$ if possible.

**Example 16**: Let $f$ be a function defined by $f(x) = -x^2 - 1$ and let $g$ be a function defined by $g(x) = \begin{cases} 1 - x & , x < 0 \\ x^2 + 1 & , x \ge 0 \end{cases}$. Find $g \circ f$ if possible.

## 3 Types of Functions

In this section, we will be introduced several types of functions. In general, functions have two types: algebraic functions and transcendental functions.

3.1 **Algebraic functions**

There are two types of algebraic functions.

1) **Polynomial functions**: Polynomial function is a function of the form $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0$ where $n$ is a positive integer or zero and $a_0, a_1, a_2, ..., a_n$ are real numbers. Since the polynomial function is a function from a set of real numbers to real numbers, therefore the domain of polynomial functions is a set of real numbers.

2) **Rational functions**: Rational function is a function of the form $y = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions and $Q(x) \neq 0$. For example, $\frac{x^2+1}{x^2-x+2}, \frac{x-1}{2x^2+x+1}, \frac{1}{x-3}$,etc. The domain of a rational function is a set of any real numbers where $Q(x) \neq 0$, that is $D_f = \{x \in \mathbb{R} | Q(x) \neq 0\}$.

3.2 **Transcendental functions**

Transcendental function is a function that is not an algebraic function.

1) **Exponential functions base *a***: An exponential function is the function of the form $y = ba^{P(x)}$ where $a$ and $b$ are real numbers, $a > 0$, $a \neq 1$, $b \neq 0$, and $P(x)$ is a polynomial function. For example, $y = 2^{x+3}$, $y = -3^{x^2+1}$, $y = e^{2x}$, etc. The domain of an exponential function base *a* is a set of real numbers.

2) **Logarithmic functions:** A logarithmic function is an inverse function of an exponential function, that is the logarithmic function that is defined by $y = \log_a x$ will be an inverse function of the exponential function base *a* which is $y = a^x$ where $a > 0$ and $a \neq 1$. For example, $y = \log_2 x$, $y = \log x$, and $y = \ln x$, etc.

**Note:** (i) Logarithmic function base 10 is denoted by $y = \log x$.

(ii) The natural logarithm is the logarithmic function base *e* and is denoted by $y = \ln x$.

# 4 Applications of Functions

**Example 1**: A rectangular box has a square base and the length of each side of the base is twice the height of the box. Let $f(x)$ be a function of its volume and let $x$ be the length of each side of the base (in centimeter).

1) Find $f(x)$.
2) If the height of this box is 6 cm, then find the volume of the box.

**Example 2**: A company produces two products called A and B. They produce product A and product B in a ratio of 1:3. The cost of producing both products is the same, which is $1-3x^2-x$ (in Baht) where $x$ is number of product A that is produced by this company. The sale manager set the price for product A and B equal to 25 Baht and 23 Baht, respectively. Assume that the company sold all the products.

1) Find the function of the profit from selling both products in term of $x$.
2) If the company produces 1,050 units of product B, find the profit from selling both products.