Unbiased Estimator for Variance of the Disturbance Terms

Unbiased estimator for σ_u^2

Since variance of the OLS estimators ($\hat{\beta}_i$) depends on the value of σ_u^2 (variance of the disturbance term u_i), we need to have an estimator for σ_u^2.

We do not have the observed values of u_i but we have the observed values of the estimated residuals $\hat{u}_i$. So we will use $\hat{u}_i$ to estimate σ_u^2

Model: $Y_i = \beta_1 + \beta_2 X_i + u_i$ ....(1)

$\bar{Y} = \beta_1 + \beta_2 \bar{X} + \bar{u}$ ......(2)

Subtract (2) from (1) we obtain

$$Y_i - \bar{Y} = \beta_2 (X_i - \bar{X}) + u_i - \bar{u}$$

or $y_i = \beta_2 x_i + (u_i - \bar{u})$ .....(3)

Recall that $y_i = \hat{y}_i + \hat{u}_i$

where $\hat{y}_i = \hat{\beta}_2 x_i$

$\hat{u}_i = y_i - \hat{\beta}_2 x_i$ .... (4)

Substitute (3) into (4)

$$\hat{u}_i = \beta_2 x_i + (u_i - \bar{u}) - \hat{\beta}_2 x_i$$

$$\hat{u}_i = (u_i - \bar{u}) - (\hat{\beta}_2 - \beta_2) x_i$$

$$\hat{u}_i = (u_i - \bar{u}) - (\hat{\beta}_2 - \beta_2)\, x_i$$

$$\hat{u}_i^2 = (u_i - \bar{u})^2 + (\hat{\beta}_2 - \beta_2)^2 x_i^2 - 2(\hat{\beta}_2 - \beta_2)\, x_i (u_i - \bar{u})$$

$$\sum \hat{u}_i^2 = \sum (u_i - \bar{u})^2 + (\hat{\beta}_2 - \beta_2)^2 \sum x_i^2$$

$$- 2(\hat{\beta}_2 - \beta_2) \sum x_i (u_i - \bar{u}).$$

$$E\left[\sum \hat{u}_i^2\right] = E\left[\sum (u_i - \bar{u})^2\right] + E\left[(\hat{\beta}_2 - \beta_2)^2 \sum x_i^2\right]$$

$$- 2E\left[(\hat{\beta}_2 - \beta_2) \sum x_i (u_i - \bar{u})\right]$$

$$E\left[\sum (u_i - \bar{u})^2\right] = (n-1)\sigma_u^2$$

$$E(\hat{\beta}_2 - \beta_2)^2 \sum x_i^2 = \text{var}\,\hat{\beta}_2 . \sum x_i^2$$

$$= \frac{\sigma_u^2}{\sum x_i^2} . \sum x_i^2$$

$$= \sigma_u^2$$

$$-2E\left[(\hat{\beta}_2 - \beta_2)\sum x_i (u_i - \bar{u})\right] = -2\sigma_u^2$$

Therefore

$$E\left[\sum \hat{u}_i^2\right] = (n-1)\sigma_u^2 + \sigma_u^2 - 2\sigma_u^2$$

$$= n\sigma_u^2 - \sigma_u^2 + \sigma_u^2 - 2\sigma_u^2$$

$$= (n-2)\sigma_u^2 .$$

If we define $\hat{\sigma}_u^2 = \dfrac{\sum \hat{u}_i^2}{n-2}$

then its expected value is

$$E(\hat{\sigma}_u^2) = \frac{1}{n-2} E(\sum \hat{u}_i^2) = \sigma_u^2$$

which shows that $\hat{\sigma}_u^2$ is an unbiased estimator for σ_u^2.