Macroeconomics

Lecture 12

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The Term Structure of Interest Rate

- What is the relationship between the level of an equilibrium risk free interest rate and its term to maturity?

The Term Structure of Interest Rate

- Suppose that there are markets in one-period and two-period perfectly safe loans.
- R_{1t} and R_{2t} are risk free gross rate of return from the one- and two-period safe loans, respectively. R_{1t} and R_{2t} are known with certainty at time t.
- $\therefore R_{1t}^{-1}$ is the price of a perfectly sure claim to 1 unit of consumption at time t+1.

$$Max \qquad E_0 \sum_{t=0}^{\infty} \beta^t U(c_t),$$

$$s.t. \qquad c_t + L_{1t} + L_{2t} \le d_t + L_{1t-1} R_{1t-1} + L_{2t-2} R_{2t-2},$$

where L_{jt} *is the amount lent for* j *periods at time* t.

The Lagrangian function is,

$$J = E_0 \sum_{t=0}^{\infty} \beta^t \left[U(c_t) + \lambda_t \left(d_t + L_{1t-1} R_{1t-1} + L_{2t-2} R_{2t-2} - c_t - L_{1t} - L_{2t} \right) \right]$$

The first-order necessary conditions are

$$\frac{\partial J}{\partial c_t} = 0, \Rightarrow \qquad \left[\partial U(c_t) / \partial c_t \right] - \lambda_t = 0,$$

$$\frac{\partial J}{\partial L_{1t}} = 0, \Rightarrow \qquad -\lambda_t + \beta E_t \lambda_{t+1} R_{1t} = 0,$$

$$\frac{\partial J}{\partial L_{2t}} = 0, \Rightarrow \qquad -\lambda_t + \beta^2 E_t \lambda_{t+2} R_{2t} = 0,$$

$Substituting\ the\ first\ into\ each\ of\ the\ second$
$and\ third\ conditions\ gives$

$$E_t\left[\beta\frac{\left(\partial U\left(c_{t+1}\right)/\partial c_{t+1}\right)}{\left(\partial U\left(c_t\right)/\partial c_t\right)}R_{1t}\right]=1, \qquad (1)$$

$$E_t\left[\beta^2\frac{\left(\partial U\left(c_{t+2}\right)/\partial c_{t+2}\right)}{\left(\partial U\left(c_t\right)/\partial c_t\right)}R_{2t}\right]=1. \qquad (2)$$

$Since\ R_{1t}\ and\ R_{2t}\ are\ known\ with\ certainty\ at\ the\ beginning\ of\ t,$

$$\therefore R_{1t}^{-1}=E_t\beta\frac{\left(\partial U\left(c_{t+1}\right)/\partial c_{t+1}\right)}{\left(\partial U\left(c_t\right)/\partial c_t\right)}=the\ price\ of\ a\ perfectly\ sure\ claim\ to$$

$$1\ unit\ of\ consumption\ at\ time\ t+1.$$

$This\ price\ can\ be\ easily\ obtained\ from\ the\ following\ reasoning,$

$Since\ R_{1t}\ units\ of\ consumption\ at\ t+1\ =\ 1\ unit\ of\ consumption\ at\ time\ t$

$hence,\ 1\ "\ "\ "\ =R_{1t}^{-1}\ "\ "\ "$

$$R_{2t}^{-1}=E_t\beta^2\frac{\left(\partial U\left(c_{t+2}\right)/\partial c_{t+2}\right)}{\left(\partial U\left(c_t\right)/\partial c_t\right)}=the\ price\ of\ a\ perfectly\ sure\ claim\ to$$

$$1\ unit\ of\ consumption\ at\ time\ t+2.$$

Let $U(c) = \ln c$ *and set* $c_t = d_t$, *Eq* (1) *and* (2) *become*

$$R_{1t}^{-1} = \beta E_t \left(d_t / d_{t+1} \right)$$

$$R_{2t}^{-1} = \beta^2 E_t \left(d_t / d_{t+2} \right)$$

Suppose that $d_{t+1} = \rho d_t \theta_{t+1}, \qquad \rho > 0,$

where θ_{t+1} *is a sequence of i.i.d. random variables that are positive with probability one. Then*

$$R_{1t}^{-1} = \frac{\beta}{\rho}\left[E\left(\frac{1}{\theta} \right) \right], \quad or \quad R_{1t} = \frac{\rho}{\beta E\left(\theta^{-1}\right)},$$

$$R_{2t}^{-1} = \left(\frac{\beta}{\rho} \right)^2 \left[E\left(\frac{1}{\theta} \right) \right]^2, \quad or \quad R_{2t} = \left(\frac{\rho}{\beta E\left(\theta^{-1}\right)} \right)^2.$$

The level of interest rates rises (or falls) with the term to maturity if

$$\frac{\rho}{\left[\beta E\left(\theta^{-1}\right)\right]} > 1, \left(or \quad \frac{\rho}{\left[\beta E\left(\theta^{-1}\right)\right]} < 1 \right).$$

$$If\ \ R_{1t} = (1+r) = \frac{\rho}{\beta E(\theta^{-1})} > 1,\ then\ \ r > 0.$$

$$It\ also\ \ follows\ that\ \ R_{2t} = \left(\frac{\rho}{\beta E(\theta^{-1})}\right)^2 = (1+r)^2 > R_{1t},$$

$$(The\ \ level\ of\ \ interest\ rates\ rises\ \ with\ the\ \ term\ \ to\ \ maturity)$$

R_{1t} *and* R_{2t} *are risk free gross rates of return becasue they are the rates of return of perfectly safe loans.*

The values of R_{1t} *and* R_{2t} *are obatined from the equilibrium of the economy with uncertain future dividends.*

$$Let\ us\ write\ Eq(1)\ as$$

$$\beta \frac{\left(\partial U\left(c_{t+1}\right)/\partial c_{t+1}\right)}{\left(\partial U\left(c_{t}\right)/\partial c_{t}\right)} R_{1t} = 1 + \varepsilon_{1t+1},$$

$$where\ E_t \varepsilon_{1t+1}\ is\ the\ least\ squares\ residual\ \ and\ \ E_t \varepsilon_{1t+1} = 0.$$

$$\therefore\ \beta \frac{\left(\partial U\left(c_{t+1}\right)/\partial c_{t+1}\right)}{\left(\partial U\left(c_{t}\right)/\partial c_{t}\right)} = R_{1t}^{-1} + R_{1t}^{-1} \varepsilon_{1t+1} \qquad (3)$$

$$Eq\ (2)\ can\ be\ written\ as$$

$$R_{2t}^{-1} = E_t \left[\beta^2 \frac{\left(\partial U\left(c_{t+1}\right)/\partial c_{t+1}\right)}{\left(\partial U\left(c_{t}\right)/\partial c_{t}\right)} \frac{\left(\partial U\left(c_{t+2}\right)/\partial c_{t+2}\right)}{\left(\partial U\left(c_{t+1}\right)/\partial c_{t+1}\right)} \right] \qquad (4)$$

$$Substituting\ Eq\ (3)\ \text{int}\,o\ Eq\ (4)\ gives$$

$$R_{2t}^{-1} = E_t \left[\left(R_{1t}^{-1} + R_{1t}^{-1} \varepsilon_{1t+1} \right) \left(R_{1t+1}^{-1} + R_{1t+1}^{-1} \varepsilon_{1t+2} \right) \right]$$

$$= R_{1t}^{-1} E_t \left[R_{1t+1}^{-1} + R_{1t+1}^{-1} \varepsilon_{1t+1} + R_{1t+1}^{-1} \varepsilon_{1t+1} \varepsilon_{1t+2} + R_{1t+1}^{-1} \varepsilon_{1t+2} \right].$$

$$Or, \qquad R_{2t}^{-1} = R_{1t}^{-1} \left[E_t R_{1t+1}^{-1} + E_t R_{1,t+1}^{-1} E \varepsilon_{t_{1t+1}} + \text{cov}_t \left(R_{1t+1}^{-1}, \varepsilon_{1t+1} \right) \right]$$

$$+ R_{1t}^{-1} E_t R_{1t+1}^{-1} \varepsilon_{1t+1} \varepsilon_{1t+2} + R_{1t}^{-1} E_t R_{1t+1}^{-1} \varepsilon_{1t+2} \qquad (5)$$

Consider the last two terms

$$E_t R_{1t+1}^{-1} \varepsilon_{1t+1} \varepsilon_{1t+2} = E_t E_{t+1} R_{1t+1}^{-1} \varepsilon_{1t+1} \varepsilon_{1t+2}$$

$$= E_t R_{1t+1}^{-1} E_{t+1} \varepsilon_{1t+1} \varepsilon_{1t+2}$$

$$= E_t R_{1t+1}^{-1} E_{t+1} \varepsilon_{1t+1} E_{t+1} \varepsilon_{1t+2} = 0,$$

$$E_t R_{1t+1}^{-1} \varepsilon_{1t+2} = E_t E_{t+1} R_{1t+1}^{-1} \varepsilon_{1t+2} = E_t R_{1t+1}^{-1} E_{t+1} \varepsilon_{1t+2} = 0.$$

Substituting these 2 equalities into (5) gives

$$R_{2t}^{-1} = R_{1t}^{-1} \left[E_t R_{1t+1}^{-1} + \operatorname{cov}_t \left(R_{1t+1}^{-1}, \varepsilon_{1t+1} \right) \right], \qquad (6)$$

where $$\varepsilon_{1t+1} = \beta \frac{\left(\partial U \left(c_{t+1} \right) / \partial c_{t+1} \right)}{\left(\partial U \left(c_t \right) / \partial c_t \right)} R_{1t} - 1.$$

For example,

$$\uparrow R_{1t+1}^{-1} \Rightarrow \downarrow c_{t+2}, \uparrow c_{t+1} \Rightarrow \downarrow \frac{\partial U \left(c_{t+1} \right)}{\partial c_{t+1}} \left(\text{since utility is concave} \right) \Rightarrow \downarrow \varepsilon_{1t+1},$$

hence, $\operatorname{cov}_t \left(R_{1t+1}^{-1}, \varepsilon_{1t+1} \right) < 0,$ *so from* (6),

$$R_{1t} = R_{2t} \cdot \left[E_t R_{1t+1}^{-1} + \operatorname{cov}_t \left(R_{1t+1}^{-1}, \varepsilon_{1t+1} \right) \right],$$

$$R_{1t} E_t R_{1t+1} = R_{2t} + \operatorname{cov}_t \left(R_{1t+1}^{-1}, \varepsilon_{1t+1} \right) R_{2t} E_t R_{1t+1} < R_{2t}$$

$Eq.(6)$ is a generalized version of the pure expectation theory of the term structure of interest rates, adjusted for the risk premium $\operatorname{cov}_t\left(R_{1t+1}^{-1},\varepsilon_{1t+1}\right)$.

If the private agent is risk aversive, then $\operatorname{cov}_t\left(R_{1t+1}^{-1},\varepsilon_{1t+1}\right)<0$.

If the private agent is risk lover, then $\operatorname{cov}_t\left(R_{1t+1}^{-1},\varepsilon_{1t+1}\right)>0$.

The pure expectation theory states that $R_{2t}^{-1}=R_{1t}^{-1}\left[E_t R_{1t+1}^{-1}\right]$, this is the result from the assumption that private agent is risk neutral.

Contingent Claim Market

Pricing of one-step-ahead state-contingent securities.

- **Let the state of the economy evolve according to a Markov process described by density f(x',x), hence,**

$$prob(x_{t+1} \leq x' \mid x_t = x) = \int_{-\infty}^{x'} f(u,x)\,du \equiv F(x',x)$$

Contingent Claim Market

- In period t, given that the economy is in state x_t, then one can purchase, or sell, a claim to 1 unit of next period consumption good contingent on the event that x_{t+1} belongs to a set A, at the following price (measured in terms of time t consumption)

$$\int_{x_{t+1} \in A} q\left(x_{t+1}, x_t\right) dx_{t+1}$$

Contingent Claim Market

- **Let Ω be the entire space of possible x, the time t, price of a perfectly certain claim on period (t+1) consumption is**

$$\int_{x_{t+1}\in\Omega} q\left(x_{t+1}, x_t\right) dx_{t+1} = \frac{1}{R_{1t}}$$

$$= the\ \ price\ of\ \ a\ \ perfectly\ sure\ claim\ to$$
$$1\ unit\ of\ \ consumption\ at\ time\ t+1,$$

where

$$\int_{x_{t+1}\in\Omega} q\left(x_{t+1}, x_t\right) dx_{t+1} > \int_{x_{t+1}\in A} q\left(x_{t+1}, x_t\right) dx_{t+1}.$$