Partial Fraction Decomposition

1 Partial-fraction decomposition

Definition 1.1 (Polynomial)**.** A polynomial $p(x)$ of degree n is a function in the form of

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

where $a_0, a_1, \ldots, a_n$ are some constants.

Partial-fraction decomposition is a useful tool for computing integration of functions in the form

$$\frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials with the degree (highest power of x) of $p(x)$ is smaller than the degree of $q(x)$. I.e.$\frac{p(x)}{q(x)}$ has to be a "proper" fraction.

- Partial-fraction decomposition only works for "proper" fractions. That is, if the denominator's degree is not larger than the numerator's degree (so you have, in effect, an "improper" polynomial fraction), then you first have to use long division to get the "mixed number" form of the rational expression. Then decompose the remaining fractional part.
- Partial fraction decomposition will be in the form of the sum of the *proper fractions* of whose denominators are polynomials of degree either 1 or "irreducible" polynomials of degree 2.

Definition 1.2. (Proper fraction & Irreducible polynomial)

- A rational function $\frac{p(x)}{q(x)}$ is called a **proper fraction** if $p(x)$ and $q(x)$ are polynomials with the degree (highest power of x) of $p(x)$ is smaller than the degree of $q(x)$.
- A polynomial is **irreducible** if it is a non-constant polynomial that may not be factored into the product of two non-constant polynomials with lower degree.
  E.g.$x - 3$, $x^2 + 1$, $x^2 + x + 1$.

Example 1.1. Determine if each of the following rational functions is a proper fraction.

- $\frac{x^2}{x+1}$
- $\frac{x+100}{x^2+1}$
- $\frac{x^3+1}{4-x^3}$

2 Procedure for Partial-fraction decomposition

The process of partial fraction starts from factoring the denominator into polynomials of degree either 1 or "irreducible" polynomials of degree 2.

1. Factor of the form $ax + b$, $a \neq 0$. Partial fraction decomposition:

$$\frac{A}{ax+b}$$

2. Factor of the form $(ax + b)^n$, $a \neq 0$, $n \geq 2$. Partial fraction decomposition:

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \dots \frac{A_n}{(ax+b)^n}.$$

3. Factor of the form $ax^2 + bx + c$, $a \neq 0$ (with $b^2 - 4ac < 0$). Partial fraction:

$$\frac{Ax+B}{ax^2+bx+c}$$

4. Factor of the form $(ax^2+bx+c)^n$, $a \neq 0$, $n \geq 2$(with $b^2-4ac < 0$). Partial fraction decomposition:

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \frac{A_3x+B_3}{(ax^2+bx+c)^3} + \dots \frac{A_nx+B_n}{(ax^2+bx+c)^n}.$$

The constants A, A_i, B_i above can be found by using one of the following techniques.

1. Compare the coefficients of the same degrees for x.
2. Substitute the certain values for x.

Example 2.1. Find the partial fraction decomposition for $\dfrac{6}{(x-1)(x+1)}$.

Example 2.2. Find the partial fractions decomposition for

$$\frac{2x^4 + 7x^2 + 4}{x^4 + x^2}.$$

Example 2.3. Find the partial fraction decomposition for

$$\frac{3x^5 - x^4 + 7x^3 - 3x^2 + 3x - 2}{x^6 + 2x^4 + x^2}.$$

Example 2.4. Find the partial fraction decomposition for

$$\frac{x^2+4}{3x^3+4x^2-4x}.$$