Quiz 2: Date: May 5, 2022 from 11.00-12.30

Question 1 (40 marks) Score.....................

Consider the Muliperiod model of consumption and portfolio choice. Let an individual in this economy has the utility function as follow:

$$\max_{C_s,\omega_s,\forall_t} E_t\left[\sum_{s=t}^{T-1}\delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)+\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

Assume that there is no wage income $(y_t = 0\,\forall t)$ and a constant risk-free rate return asset , $R_{ft} = R_f$. Also asssume that n=1 and the return of a single risky asset, R_{rt},is independently and identically distributed over time. Denote the proportion of wealth invested in the risky asset at date t as ω_t.

Please read and answer the following questions carefully and completely.

Score.....................

Question 1.1 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-1, C^*_{T-1} and ω^*_{T-1}., and give an explicit expression for C^*_{T-1}

$$u_c(C_{T-1}, T-1) = R_{F,T-1} E_{T-1}[B_W(W_T, T)]$$

$$\frac{\delta^{T-1}}{C^*_{T-1}} = R_{F,T-1} E_{T-1}\left[\delta^T \frac{1}{S_{T-1} R_{T-1}}\right]$$

$$1 = \frac{\delta C^*_{T-1} R_{F,T-1}}{W_{T-1} - C^*_{T-1}} E_{T-1}\left[\frac{1}{R_{T-1}}\right]$$

$$J(W_{T-1}, T-1) = \delta^{T-1} \ln[C^*_{T-1} + \delta^T E_{T-1}[\ln[R^*_{T-1}(W_{T-1} - C^*_{T-1})]]]$$

$$= \delta^{T-1}(-\ln[1+\delta] + \ln[W_{T-1}]) + \delta^T\left(E_{T-1}[\ln[R^*_{T-1}]] + \ln\left[\frac{\delta}{1+\delta}\right] + \ln[W_{T-1}]\right)$$

Score.....................

Question 1.2 (10 marks) Solve for the form of J(W_{T-1},T-1).

From $J(W_t, I_t, t) = \max E_t\left[\sum_{s=t}^{T-1} u(c_s, s) + B(W_T, T)\right]$

At date T-1,

$$J(W_{T-1}, T-1) = \max E_{T-1}\left[u(c_{T-1}, T-1) + B(W_T, T)\right]$$

$$= \max_{c_{T-1}, \{w_1, T-1\}} u(c_{T-1}, T-1) + E_{T-1}\left[B(W_T, T)\right]$$

$$= \max u(c_{T-1}, T-1) + E_{T-1}\left[B(S_{T-1} R_{T-1}, T)\right]$$

set the result = 0 ; $u_c(c_{T-1}, T-1) - E_{T-1}\left[B_W(W_T, T) R_{T-1}\right] = 0$

$$E_{T-1}\left[B_W(W_T, T)(R_{1,T-1} - R_{f,T-1})\right] = 0$$

$$u_c(c_{T-1}, T-1) = E_T\left[B_W(W_T, T)(R_{1,T-1} + \omega_{1,T-1}(R_{1,T-1} - R_{f,T-1})\right]$$

$$= R_{f,T-1} E_{T-1}\left[B_W(W_T, T)\right]$$

c^*_{T-1} and $\omega^*_{i,T-1}$; $J_W = u_c \dfrac{\partial c^*_{T-1}}{\partial W_{T-1}} + E_{T-1}\left(B_{W_T} \cdot \dfrac{dW_T}{dW_{T-1}}\right)$

$$= u_c \frac{\partial c^*_{T-1}}{\partial W_{T-1}} + E_{T-1}\left[B_{W_T} \cdot \left[R_{1,T-1} - R_{f,T-1}\right] S_{T-1} \frac{\partial \omega^*_{1,T-1}}{\partial W_{T-1}} + R_{T-1}\left(1 - \frac{\partial c^*_{T-1}}{\partial W_{T-1}}\right)\right]$$

FOC. $J_W = u_c \dfrac{\partial c^*_{T-1}}{\partial W_{T-1}} - E_{T-1}\left[B_{W_T} R_{T-1}\right] \dfrac{\partial c^*_{T-1}}{\partial W_{T-1}} + E_{T-1}\left[B_{W_T} R_{T-1}\right]$

$$= E_{T-1}\left[B_{W_T} R_{T-1}\right]$$

So $J_W(W_{T-1}, T-1) = u_c(c^*_{T-1}, T-1)$

Score.....................

Question 1.3 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-2, C^*_{T-2} and ω^*_{T-2}., and give an explicit expression for C^*_{T-2}

$$u_c(C^*_{T-2}, T-1) = E_{T-2}\left[J_w(W_{T-1}, T-1) R_{T-2}\right]$$

$$\frac{\delta^{T-2}}{C_{T-2}} = (1+\delta)\delta^{T-1} E_{T-2}\left[\frac{R_{T-2}}{S_{T-2} R_{T-2}}\right]$$

$$= \frac{(1+\delta)\delta^{T-1}}{W_{T-2} - C_{T-2}}$$

$$C^*_{T-2} = \frac{1}{1+\delta+\delta^2} W_{T-2}$$

$$\frac{\delta^{T-2}}{C^*_{T-2}} = R_{F,T-2}\,\delta^{T-1} E_{T-2}\left[\frac{1+\delta}{S_{T-2} R_{T-2}}\right]$$

$$1 = \frac{\delta(1+\delta) C^*_{T-2} R_{F,T-2}}{W_{T-2} - C^*_{T-2}} E_{T-2}\left[\frac{1}{R_{T-2}}\right]$$

$$1 = R_{F,T-2} E_{T-2}\left[\frac{1}{R_{T-2}}\right]$$

Score.....................

Question 1.4 (10 marks) Solve for the form of J(W_{T-2},T-2).Based on the pattern for T-1 and T-2, provide expressions for the optimal consumption and portfolio weight at any date T-t, t=1,2,3,....

$$J(W_{T-2}, T-2) = \max_{c_{T-2}, \{\omega_{1,T-2}\}} u(c_{T-2}, T-2) + E_{T-2}\left[u(c_{T-1}, T-1) + B(W_T, T)\right]$$

From $\max_{\{(T-2),(T-1)\}} (y) = \max_{\{T-2\}} \left\{ \max_{\{T-1 \mid (T-2)\}} (y) \right\}$

so $J(W_{T-2}, T-2) = \max_{c_{T-2}, \{\omega_{1,T-2}\}} \left\{ u(c_{T-2}, T-2) + E_{T-2}\left[\max_{c_{T-1}, \{\omega_{1,T-1}\}} E_{T-1}\left[u(c_{T-1}, T-1) + B(W_T, T)\right]\right] \right\}$

$$= \max_{c_{T-2}, \{\omega_{1,T-2}\}} u(c_{T-2}, T-2) + E_{T-2}\left[J(W_{T-1}, T-1)\right]$$