EE320 Chapter 4
Linear Model, Basic Matrix Algebra and Application

Panit Wattanakoon
Faculty of Economics, Thammasat University

1 Terminology and Type of Matrix

1.1 Definition

$$\mathbb{A}_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

Subscript of each a_{ij} indicates dimension of particular element:

m $\times$ n = row $\times$ column

Example $\mathbb{B}_{2\times 2} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

$b_{11} = 1$
$b_{12} = 2$
$b_{21} = 3$
$b_{22} = 4$

1.2 Types of matrix

1. Square Matrix : number of rows = number of columns

$$\text{ex. } \mathbb{A}_{3\times 3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

2. Identity Matrix: square matrix with diagonal elements being equal to 1 and the rest equal to 0

$$\text{ex. } \mathbb{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\text{diagonal elements} = 1}$$

3. Column Vector: $\mathbb{A} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1\times 3}$ when n=1

4. Row Vector: $\mathbb{B} = \begin{bmatrix} 4 \\ 3 \\ 6 \end{bmatrix}_{1\times 3}$ when m =1

2 Matrix Operation

2.1 Basic Operation

1. $\mathbb{A} = \mathbb{B} \quad \text{iff} \quad a_{ij} = b_{ij} \quad \forall i \forall j$

2. Suppose we have matrices of $\mathbb{A}$ and $\mathbb{B}$ as follows:

$$\left.\begin{aligned} \mathbb{A} &= \begin{bmatrix} a_{ij} \end{bmatrix}_{m\times n} \\ \mathbb{B} &= \begin{bmatrix} b_{ij} \end{bmatrix}_{m\times n} \end{aligned}\right\} \text{same dimension}$$

Addition and subtraction: $\mathbb{A} \pm \mathbb{B} = \begin{bmatrix} a_{ij} \end{bmatrix}_{m\times n} \pm \begin{bmatrix} b_{ij} \end{bmatrix}_{m\times n} = \begin{bmatrix} a_{ij} \pm b_{ij} \end{bmatrix}_{m\times n}$

$\underline{\text{ex.}}$ $\mathbb{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\qquad$ $\mathbb{A} + \mathbb{B} = \begin{bmatrix} 5 & 5 \\ 5 & 4 \end{bmatrix}$

$\mathbb{B} = \begin{bmatrix} 4 & 4 \\ 4 & 3 \end{bmatrix}$ $\qquad$ $\mathbb{A} - \mathbb{B} = \begin{bmatrix} -3 & -3 \\ -3 & -2 \end{bmatrix}$

3. If α is a real number, then $\alpha\mathbb{A} = \alpha \left[a_{ij}\right]_{m\times n} = \left[\alpha a_{ij}\right]_{m\times n}$

   $\underline{\text{ex.}}$ $5\mathbb{A} =$

4. Rules of matrix addition and multiplication by constants:

$$\begin{array}{lll} 1) & (\mathbb{A}+\mathbb{B})+\mathbb{C} & = \mathbb{A}+(\mathbb{B}+\mathbb{C}) \\ 2) & \mathbb{A}+\mathbb{B} & = \mathbb{B}+\mathbb{A} \\ 3) & \mathbb{A}+\underline{0} & = \mathbb{A} \\ 4) & \mathbb{A}+(-\mathbb{A}) & = \underline{0} \\ 5) & (\alpha+\beta)\mathbb{A} & = \alpha\mathbb{A}+\beta\mathbb{A} \\ 6) & \alpha(\mathbb{A}+\mathbb{B}) & = \alpha\mathbb{A}+\alpha\mathbb{B} \end{array}$$

2.2 Matrix Multiplication

Suppose $\mathbb{A} = \left[a_{ij}\right]_{m\times \underline{n}}$ and $\mathbb{B} = \left[b_{ij}\right]_{\underline{n}\times p}$, then the product $\mathbb{C} = \mathbb{A}\mathbb{B} = \left[c_{ij}\right]_{m\times p}$ and each element in $\mathbb{C}$ is as follows:

$$c_{ij} = \sum_{r=1}^{n} a_{ir}b_{rj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

$\underline{\text{ex.}}$ $\mathbb{A}_{1\times 2} = \begin{bmatrix} a_{11} & a_{12} \end{bmatrix}$ $\mathbb{B}_{2\times 3} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$

$$\begin{aligned} \mathbb{C}_{1\times 3} = \mathbb{A}_{1\times \underline{2}}\mathbb{B}_{\underline{2}\times 3} &= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} & a_{11}b_{13}+a_{12}b_{23} \end{bmatrix}_{1\times 3} \\ &= \begin{bmatrix} c_{11} & c_{12} & c_{13} \end{bmatrix} \end{aligned}$$

But $\mathbb{D} = \mathbb{B}\cdot\mathbb{A}$ not possible since column of $\mathbb{B} \neq$ row of $\mathbb{A}$

ex. $\mathbb{A} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{2\times 1}$ $\mathbb{B} = \begin{bmatrix} 4 & 3 \end{bmatrix}_{1\times 2}$

$\mathbb{C}_{2\times 2} = \mathbb{A}_{2\times 1} \cdot \mathbb{B}_{1\times 2} =$

ex. $\mathbb{A} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}_{2\times 2}$ $\mathbb{B} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}_{2\times 2}$

$\mathbb{C}_{2\times 2} = \mathbb{A}_{2\times 2} \cdot \mathbb{B}_{2\times 2} =$

ex. $\mathbb{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\mathbb{A}\mathbb{I}_2 = \begin{bmatrix} & \\ & \end{bmatrix}$

$\mathbb{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\mathbb{I}_2\mathbb{A} = \begin{bmatrix} & \\ & \end{bmatrix}$

$\therefore \mathbb{A}\mathbb{I} = \mathbb{I}\mathbb{A} = \mathbb{A}$ but $\mathbb{A}\mathbb{B} \neq \mathbb{B}\mathbb{A}$

Identity matrix is the matrix such that when it is multiplied by the other matrix, the result is that other matrix multiplied.

Rules for matrix multiplication

1) $\mathbb{A}\mathbb{B} \neq \mathbb{B}\mathbb{A}$ in general

2) $(\mathbb{A}\mathbb{B})\mathbb{C} = \mathbb{A}(\mathbb{B}\mathbb{C})$

3) $(\mathbb{A} + \mathbb{B})\mathbb{C} = \mathbb{A}\mathbb{C} + \mathbb{B}\mathbb{C}$

2.3 Matrix Transposition

$\mathbb{A} = \left[a_{ij}\right]_{m\times n}$

$\mathbb{A}' = \mathbb{A}^{\mathrm{T}} = \left[a_{ij}\right]_{n\times m}$

<u>ex.</u> $\mathbb{A} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{2\times 1} \quad \to \mathbb{A}^{\mathrm{T}} =$

<u>ex.</u> $\mathbb{B} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2} \to \mathbb{B}^{\mathrm{T}} =$

<u>Rules for transposes</u>

$$\begin{array}{lll} 1) \ (\mathbb{A}^{\mathrm{T}})^{\mathrm{T}} & = & \mathbb{A} \\ 2) \ (\mathbb{A}+\mathbb{B})^{\mathrm{T}} & = & \mathbb{A}^{\mathrm{T}}+\mathbb{B}^{\mathrm{T}} \\ 3) \ (\mathbb{A}\mathbb{B})^{\mathrm{T}} & = & \mathbb{B}^{\mathrm{T}}\mathbb{A}^{\mathrm{T}} \\ 4) \ (\alpha\mathbb{A})^{\mathrm{T}} & = & \alpha\mathbb{A}^{\mathrm{T}} \end{array}$$

<u>ex.</u> $\quad \mathbb{A} = \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix} \quad \mathbb{B} = \begin{bmatrix} -5 & 2 \\ 6 & 0 \end{bmatrix}$

$\mathbb{A}\mathbb{B} =$

$(\mathbb{A}\mathbb{B})^{\mathrm{T}} =$

$\mathbb{B}^{\mathrm{T}}\mathbb{A}^{\mathrm{T}} =$

3 System of Linear Equation

$$\begin{aligned}3x_1 + 4x_2 &= 5\\ 7x_1 + 4x_2 &= 2\end{aligned}$$

We can write the above two equations and two unknowns in matrix form:

$$\mathbb{AX} = b$$

$$\underbrace{\begin{bmatrix}3 & 4\\ 7 & 4\end{bmatrix}_{2\times2}}_{\mathbb{A}} \underbrace{\begin{bmatrix}x_1\\ x_2\end{bmatrix}_{2\times1}}_{\mathbb{X}} = \underbrace{\begin{bmatrix}5\\ 2\end{bmatrix}}_{b}$$

$$\Leftrightarrow \begin{bmatrix}3x_1 + 4x_2\\ 7x_1 + 4x_2\end{bmatrix} = \begin{bmatrix}5\\ 2\end{bmatrix}$$

So, if we have m equations and n unknowns :

$$\left.\begin{aligned}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \text{—①}\\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \text{—②}\\ &\vdots\\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \text{—ⓜ}\end{aligned}\right\}$$

$$\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix} \begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix} = \begin{bmatrix}b_1\\ b_2\\ \vdots\\ b_m\end{bmatrix}$$

Consider general form of two equations with two unknowns:

$$\left.\begin{aligned}\text{①} \text{ — } a_{11}x_1 + a_{12}x_2 &= b_1\\ \text{②} \text{ — } a_{21}x_1 + a_{22}x_2 &= b_2\end{aligned}\right\} \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} \begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}b_1\\ b_2\end{bmatrix}$$

$$
\begin{aligned}
①\times a_{21} &: \; a_{11}a_{21}x_1 + a_{12}a_{21}x_2 = a_{21}b_1 \text{\_\_\_} ③ \\
②\times a_{11} &: \; a_{11}a_{21}x_1 + a_{11}a_{22}x_2 = a_{11}b_2 \text{\_\_\_} ④ \\
③ - ④ &: \; (a_{12}a_{21} - a_{11}a_{22})x_2 = a_{21}b_1 - a_{11}b_2
\end{aligned}
$$

$$
x_2^* = \frac{a_{21}b_1 - a_{11}b_2}{a_{12}a_{21} - a_{11}a_{22}}
$$
$$
x_1^* = \frac{a_{12}b_2 - a_{22}b_1}{a_{12}a_{21} - a_{11}a_{22}}
$$

<u>ex.</u>

$$
\begin{aligned}
x + y &= 10 \text{\_\_\_} ① \\
2x + y &= 5 \text{\_\_\_} ② \\
① - ② \quad -x &= 5 \\
x &= 5 \\
y &= 15
\end{aligned}
$$

or
$$
\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}
$$

$$
\begin{aligned}
\mathbb{A}\mathbb{X} &= b \\
\mathbb{A}^{-1}\mathbb{A}\mathbb{X} &= \mathbb{A}^{-1}b \\
\mathbb{X} &= \mathbb{A}^{-1}b
\end{aligned}
$$

4 Matrix Inversion

4.1 Definition

$\mathbb{A}\mathbb{A}^{-1} = \mathbb{A}^{-1}\mathbb{A} = \mathbb{I}_{\mathrm{n}}$, $\mathbb{A}^{-1}$ is defined iff $\mathbb{A}$ is square matrix

4.2 Properties

1) If $\mathbb{A}^{-1}$ exists, If $(\mathbb{A}^{-1})^{-1} = \mathbb{A}$

2) If $\mathbb{A}\mathbb{B}$ is invertible, then $(\mathbb{A}\mathbb{B})^{-1} = \mathbb{B}^{-1}\mathbb{A}^{-1}$

3) $(\mathbb{A}^{\mathbb{T}})^{-1} = (\mathbb{A}^{-1})^{\mathrm{T}}$

4) $(c\mathbb{A})^{-1} = (c^{-1}A^{-1})$ where c $\neq 0$

If a square matrix $\mathbb{A}$ has an inverse, $\mathbb{A}$ is said to be <u>non</u> singular.

If a square matrix $\mathbb{A}$ is <u>not</u> invertible, then $\mathbb{A}$ is called a singular matrix.

For system of linear equations:

$$\begin{aligned} \mathbb{A}\mathbb{X} &= b \\ \mathbb{A}^{-1}\mathbb{A}\mathbb{X} &= \mathbb{A}^{-1}b \\ \mathbb{X} &= \mathbb{A}^{-1}b \end{aligned}$$

5 Determinant

To test whether matrix is invertible, we can use determinant. Determinant of square matrix $\mathbb{A}$, $|A|$, is a uniquely defined number associated with that matrix.

Example:

1. $|A_1| = |\ a\ | = a$

2. $|A_2| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

3. $|A_3| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

   ① Laplace Expansion: $a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

   ② Sarrus's Law: $\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{vmatrix}$

$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$

Laplace Expansion

Let $\mathbb{A}$ be an n $\times$ n matrix.

- The "minor" of the element a_{ij} is

$$|\mathrm{\mathbb{M}_{ij}}| = \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j} & a_{1,j+1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{i,j-1} & a_{i,j} & a_{i,j+1} & \cdots & a_{i,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j} & a_{n,j+1} & \cdots & a_{n,n} \end{vmatrix}$$

- The "cofactor" of the element a_{ij} is $|C_{ij}| = (-1)^{i+j}|\mathbb{M}_{ij}|$

$|\mathbb{A}| = \sum_{j=1}^{n} a_{ij} \mid C_{ij} \mid$ expansion by the i^{th} row.

$= \sum_{i=1}^{n} a_{ij} \mid C_{ij} \mid$ expansion by the j^{th} column.

<u>ex.</u> $\mathbb{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

1^{st}row : $|A| = a_{11}|c_{11}| + a_{12}|c_{12}| + a_{13}|c_{13}|$

1^{st}column : $|A| = a_{11}|c_{11}| + a_{21}|c_{21}| + a_{31}|c_{31}|$

$$c_{11} = (-1)^{i+1}|\mathbb{M}_{11}| = (-1)^2 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$
$$c_{12} = (-1)^{i+2}|\mathbb{M}_{12}| = (-1)^3 \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$
$$c_{13} = (-1)^{i+2}|\mathbb{M}_{12}| = (-1)^4 \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

5.1 Properties of Determinants

1) $|\mathbb{A}| = |\mathbb{A}'|$:

ex.
$$|\mathbb{A}| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} =$$
$$|\mathbb{A}^{'}| = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} =$$

2) The interchange of any two rows (or columns) will alter the sign, but not numerical value of determinant.

ex.
$$|\mathbb{A}| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} =$$
$$|\mathbb{B}| = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} =$$
$$|\mathbb{C}| = \begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} =$$

3) If all elements in any row (or column) is multiplied by a scalar k, the determinant is multiplied by k.

ex. $|\mathbb{A}| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} =$

Suppose $k = 2$. For $\mathbb{B}$ and $\mathbb{C}$, multiply first row of $\mathbb{A}$ with k and first column, respectively.

$$|\mathbb{B}| = \begin{vmatrix} 2 & 4 \\ 3 & 4 \end{vmatrix} =$$
$$|\mathbb{C}| = \begin{vmatrix} 2 & 2 \\ 6 & 4 \end{vmatrix} =$$

**** 4) The addition (or substitution) of a multiple of any row to (from) another row will leave the value of the determinant unaltered.

ex. $[\mathbb{A}] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \sim \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}_{R_2 - 3R_1} \Rightarrow$

<u>Not</u> $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \sim \begin{bmatrix} & \\ & \end{bmatrix}_{2R_2 - 4R_1} \Rightarrow$

"Row Operation" *<u>not</u> column operation

<u>ex.</u> $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 \\ -1 & 4 \end{bmatrix}_{c_1 - c_2} \Rightarrow$

5) If one row (or column) is a multiple of another row (or column), the value of the determinant will be zero.

<u>ex.</u> $\mathbb{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \Rightarrow \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4 - 4 = 0$

"linearly dependent" = one row/column depends on the other.

<u>Note</u> The condition of linear <u>in</u>dependence is a sufficient condition for non singularity (invertible) of a matrix. For the rows (or column) to be linearly independent, none must be a linear combination of the other

*Nonsingularity: $|\mathbb{A}| \neq 0$

$\Leftrightarrow$ there is row (column) independence in $\mathbb{A}$

$\Leftrightarrow$ $\mathbb{A}$ is singular

$\Leftrightarrow$ $\mathbb{A}^{-1}$ exist

$\Leftrightarrow$ A unique solution $\mathbb{X}^* = \mathbb{A}^{-1}b$ exists.

6 Matrix Inversion (Computation)

If $\mathbb{A}_{n\times n}$ is a nonsingular matrix and $|\mathbb{A}| \neq 0$

The inverse of $\mathbb{A}$ is $\mathbb{A}^{-1} = \frac{1}{|\mathbb{A}|}$ adj.($\mathbb{A}$)

where $\quad \text{adj.}(\mathbb{A}) \equiv \mathbb{C}^T_{n\times n} \equiv \begin{bmatrix} \mid c_{11} \mid & \mid c_{21} \mid & \cdots & \mid c_{n1} \mid \\ \mid c_{12} \mid & \mid c_{22} \mid & \cdots & \mid c_{n2} \mid \\ \vdots & \vdots & \ddots & \vdots \\ \mid c_{1n} \mid & \mid c_{2n} \mid & \cdots & \mid c_{nn} \mid \end{bmatrix}$

and $\mid c_{ij} \mid = (-1)^{i+j} \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j} & a_{1,j+1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{i,j-1} & a_{i,j} & a_{i,j+1} & \cdots & a_{i,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j} & a_{n,j+1} & \cdots & a_{n,n} \end{vmatrix}$

ex. Find inverse of $\mathbb{A} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix}$

Step I. Find $|\mathbb{A}| = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{vmatrix} =$

Step II. Find cofactors of all elements of $\mathbb{A}$

Step III. Take transpose

$\therefore$ adj.$(\mathbb{A}) = C^T_{n\times n} =$

Step IV. $\mathbb{A}^{-1} = \frac{1}{|\mathbb{A}|} adj.(\mathbb{A}) =$

ex. Consider a partial market equilibrium of demand and supply:

$$\begin{aligned} Q^d &= Q^s \\ Q^d &= a - bP \\ Q^s &= -c + dP \end{aligned}$$

$$\Rightarrow \qquad \begin{aligned} Q^d - Q^s + 0P &= 0 \\ Q^d + 0Q^s + bP &= a \\ Q^d + Q^s - dP &= -c \end{aligned}$$

$$\Rightarrow \qquad \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix} \begin{bmatrix} Q^d \\ Q^s \\ P \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ -c \end{bmatrix}$$

7 Cramer's Rule

Given an equation system $\mathbb{A}\mathbb{X} = b$, where $\mathbb{A}$ is $n \times n$ nonsingular matrix, the solution value of j^{th} variable can be obtained from

$$\mathbb{X}_j^* = \frac{|\mathbb{A}_j|}{|\mathbb{A}|} \begin{vmatrix} a_{11} & a_{12} & \cdots & d_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & d_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & d_n & \cdots & a_{nn} \end{vmatrix}$$

where $| \mathbb{A}_j |$ is the determinant of $\mathbb{A}$ when the j^{th} column is replaced by the constant terms $d_1 , \cdots , d_n$

<u>ex.</u> $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

By Cramer's Rule:

ex.
$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix} \begin{bmatrix} Q^d \\ Q^s \\ P \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ -c \end{bmatrix}$$

8 Matrix Algebra : Application

Model 1: Specific tax on producer

$$\begin{aligned} Q^d &= Q^s \\ Q^d &= a - bP \\ Q^s &= -c + d(P - t) \end{aligned}$$

Model 2: IS-LM model

$$\begin{aligned} C &= C_0 + cY_D \\ I &= I_0 - jr \\ G &= G_0 \\ Y_D &= Y \\ Y &= C + I + G \end{aligned} \qquad \begin{aligned} M^d &= L_0 + l_1 Y - l_2 r \\ M^s &= M_0 \end{aligned}$$

Solve for IS-LM equilibrium

Equilibrium in goods market:

$$\begin{aligned} & Y &=& \; C + I + G \\ \Leftrightarrow \; & Y &=& \; C_0 + cY_D + I_0 - jr + G_0 \\ \Leftrightarrow \; & (1-c)Y &=& \; C_0 + I_0 + G_0 - j \cdot r \\ \Leftrightarrow \; & Y^* &=& \; \tfrac{1}{1-c}\Big[C_0 + I_0 + G_0 - j \cdot r\Big] \end{aligned}$$

Equilibrium in money market:

$$\begin{aligned} M^d &= M^s \\ L_0 + l_1 Y - l_2 r &= M_0 \\ l_1 Y - l_2 r &= M_0 - L_0 \end{aligned}$$

ex. Solve for endogeneous variables Y, r , I, C simultaneously.

$$\left.\begin{aligned} Y &= C + I + G_0 \\ C &= C_0 + cY \\ I &= I_0 - jr \\ M_0 &= L_0 + l_1 Y - l_2 r \end{aligned}\right\} \quad \begin{aligned} Y - C - I + 0r &= G_0 \\ -cY + C + 0I + 0r &= C_0 \\ 0Y + 0C + I + jr &= I_0 \\ l_1 Y + 0C + 0I - l_2 r &= M_0 - L_0 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 & 0 \\ -c & 1 & 0 & 0 \\ 0 & 0 & 1 & j \\ l_1 & 0 & 0 & -l_2 \end{bmatrix} \begin{bmatrix} Y \\ C \\ I \\ r \end{bmatrix} = \begin{bmatrix} G_0 \\ C_0 \\ I_0 \\ M_0 - L_0 \end{bmatrix}$$

Reference

Aroonruengsawat, Anin. EE320 Lecture Handouts.

Chiang, A. C. and Wainwright, K. (2005) "Fundamental Methods of Mathematical Economics," 4th edition, McGraw-Hill, Inc., Singapore.

Dowling, E. T. (2001) "Schaums Outline of Theory and Problem of Introduction to Mathematical Economics", 3rd edition, The McGraw-Hill Companies, Inc.

Holden, K. and Pearson, A.W. (1992) "Introductory Mathematics for Economics and Business," Second edition, The Macmillan Press Ltd.

Sydsaeter, K. and P. Hammond. (2006) "Essential Mathematics for Economic Analysis," 2nd edition, Prentice Hall.