

# System Equations Estimation Methods

After finish this session, you should understand:

- Unbiased vs Consistent vs Efficient
- The differences between Single vs System equations estimation methods
- Instrumental Variables (IV) technique
- OLS vs ILS vs 2SLS vs 3SLS
- Seemingly Unrelated Models
- Simultaneous Equations Models

# System Equations Estimation Methods

## System Equations Models

Seemingly Unrelated (SUR) Model

Simultaneous Equations Model

- Limited Information Estimation Methods  
(Single Equation Estimation Methods)
  - ILS, 2SLS, & (LIML)
- Full Information Estimation Methods  
(System Equation Estimation Methods)
  - 3SLS, I3SLS, & (FIML)

# System Equations Estimation Methods

## Finite Sample (Small Sample) Properties

**Unbiased**  $E(\hat{\beta}) = \beta$  (All sizes of sample)

## Asymptotic (Large Sample) Properties

**Consistent**  $p \lim(\hat{\beta}) = \beta$

$$\lim_{n \rightarrow \infty} \Pr(\hat{\beta} - \beta = \varepsilon) = 0 \text{ where } \varepsilon \neq 0, \varepsilon \rightarrow 0$$

(Only when  $n \rightarrow \infty$ )

## Relative Properties

**Efficient**  $\text{var}(\hat{\beta}) < \text{var}(\hat{\beta}_{other})$

(Compare with ...)

# Seemingly Unrelated Regression (SUR) Models

The models:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_m \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_m \end{bmatrix}$$

or  $y = X\beta + \varepsilon$

where

$$E[\varepsilon] = 0, \text{var}(\varepsilon) = \Sigma = \begin{bmatrix} \sigma_{11}I & \sigma_{12}I & \cdots & \sigma_{1m}I \\ \sigma_{21}I & \sigma_{22}I & \cdots & \sigma_{2m}I \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{m1}I & \sigma_{m2}I & \cdots & \sigma_{mm}I \end{bmatrix}$$

# 5 Seemingly Unrelated Regression (SUR) Models

The models:

$$\begin{bmatrix} \begin{bmatrix} y_{11} \\ y_{12} \\ \vdots \\ y_{1t_1} \end{bmatrix} \\ \begin{bmatrix} y_{21} \\ y_{22} \\ \vdots \\ y_{2t_2} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} y_{m1} \\ y_{m2} \\ \vdots \\ y_{mt_m} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & X_{111} & \cdots & X_{1k_1 1} \\ 1 & X_{112} & \cdots & X_{1k_1 2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & X_{11t_1} & \cdots & X_{1k_1 t_1} \end{bmatrix} & \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} & \cdots & \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} & \begin{bmatrix} 1 & X_{211} & \cdots & X_{2k_2 1} \\ 1 & X_{212} & \cdots & X_{2k_2 2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & X_{21t_2} & \cdots & X_{2k_2 t_2} \end{bmatrix} & \cdots & \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} & \cdots & \begin{bmatrix} 1 & X_{m11} & \cdots & X_{mk_m 1} \\ 1 & X_{m12} & \cdots & X_{mk_m 2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & X_{m1t_m} & \cdots & X_{mk_m t_m} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \beta_{10} \\ \beta_{11} \\ \vdots \\ \beta_{1k_1} \\ \beta_{20} \\ \beta_{21} \\ \vdots \\ \beta_{2k_2} \\ \vdots \\ \beta_{m0} \\ \beta_{m1} \\ \vdots \\ \beta_{mk_m} \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \vdots \\ \varepsilon_{1t_1} \end{bmatrix} \\ \begin{bmatrix} \varepsilon_{21} \\ \varepsilon_{22} \\ \vdots \\ \varepsilon_{2t_2} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} \varepsilon_{m1} \\ \varepsilon_{m2} \\ \vdots \\ \varepsilon_{mt_m} \end{bmatrix} \end{bmatrix}$$

# 6 Seemingly Unrelated Regression (SUR) Models

$$\text{var}(\varepsilon) = \Sigma = \begin{bmatrix} \begin{bmatrix} \sigma_1^2 & 0 & \dots & 0 \\ 0 & \sigma_1^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_1^2 \end{bmatrix} & \begin{bmatrix} \sigma_{12} & 0 & \dots & 0 \\ 0 & \sigma_{12} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_{12} \end{bmatrix} & \dots & \begin{bmatrix} \sigma_{1m} & 0 & \dots & 0 \\ 0 & \sigma_{1m} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_{1m} \end{bmatrix} \\ \begin{bmatrix} \sigma_{21} & 0 & \dots & 0 \\ 0 & \sigma_{21} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_{21} \end{bmatrix} & \begin{bmatrix} \sigma_2^2 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_2^2 \end{bmatrix} & \dots & \begin{bmatrix} \sigma_{2m} & 0 & \dots & 0 \\ 0 & \sigma_{2m} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_{2m} \end{bmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{bmatrix} \sigma_{m1} & 0 & \dots & 0 \\ 0 & \sigma_{m1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_{m1} \end{bmatrix} & \begin{bmatrix} \sigma_{m2} & 0 & \dots & 0 \\ 0 & \sigma_{m2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_{m2} \end{bmatrix} & \dots & \begin{bmatrix} \sigma_m^2 & 0 & \dots & 0 \\ 0 & \sigma_m^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_m^2 \end{bmatrix} \end{bmatrix}$$

# Seemingly Unrelated Regression (SUR) Models

Example: System of two equations:

$$y_{1t} = \beta_{10} + \beta_{11}x_{1t} + \beta_{12}x_{2t} + \varepsilon_{1t}$$

$$y_{2t} = \beta_{20} + \beta_{22}x_{2t} + \beta_{23}x_{3t} + \varepsilon_{2t}$$

If there exists relationship of the error terms across equations, i.e.  $E(\varepsilon_{1t}\varepsilon_{2t}) \neq 0$ .

Ignorance of this problem by using single equation estimation method (or separate OLS) will lead to less efficient estimators.

# 8 Seemingly Unrelated Regression (SUR) Models

Example: System of two equations:

$$\begin{bmatrix} \begin{bmatrix} y_{11} \\ y_{12} \\ \vdots \\ y_{1t} \end{bmatrix} \\ \begin{bmatrix} y_{21} \\ y_{22} \\ \vdots \\ y_{2t} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & x_{11} & x_{21} \\ 1 & x_{12} & x_{22} \\ \vdots & \vdots & \vdots \\ 1 & x_{1t} & x_{2t} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & x_{21} & x_{31} \\ 1 & x_{22} & x_{32} \\ \vdots & \vdots & \vdots \\ 1 & x_{2t} & x_{3t} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \beta_{10} \\ \beta_{11} \\ \beta_{12} \end{bmatrix} \\ \begin{bmatrix} \beta_{20} \\ \beta_{22} \\ \beta_{23} \end{bmatrix} \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \vdots \\ \varepsilon_{1t} \end{bmatrix} \\ \begin{bmatrix} \varepsilon_{21} \\ \varepsilon_{22} \\ \vdots \\ \varepsilon_{2t} \end{bmatrix} \end{bmatrix}$$

## 9 Seemingly Unrelated Regression (SUR) Models

If there exists relationship of the error terms across equations, i.e.  $E(\varepsilon_{1t}\varepsilon_{2t}) = \sigma_{12} = \sigma_{21} \neq 0$

$$\text{var}(\varepsilon) = \Sigma = \begin{bmatrix} \Sigma_1 & \Sigma_{12} \\ \Sigma_{21} & \Sigma_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_1^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_1^2 \end{bmatrix} & \begin{bmatrix} \sigma_{12} & 0 & \cdots & 0 \\ 0 & \sigma_{12} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{12} \end{bmatrix} \\ \begin{bmatrix} \sigma_{21} & 0 & \cdots & 0 \\ 0 & \sigma_{21} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{21} \end{bmatrix} & \begin{bmatrix} \sigma_2^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_2^2 \end{bmatrix} \end{bmatrix}$$

# Estimation of SUR Models

Two-step system estimation or FGLS

1. Estimate Var-Cov matrix  $\hat{\Sigma}$
2. Estimate all parameters  $\beta_j$  using FGLS.

$$\hat{\beta}_{SUR} = \hat{\beta}_{FGLS} = \left( X' \hat{\Sigma}^{-1} X \right)^{-1} X' \hat{\Sigma}^{-1} y$$

Then, system equation estimation method will provide a more asymptotically efficient estimators.

However, if there exists specification error problem in any equations, the problem will spread through out the whole system.

# Simultaneous Equations Model

More than one equation.

Jointly dependent or endogenous variables.

General Structural Form model can be stated as:

$$\begin{aligned}
 \gamma_{11}y_{t1} + \gamma_{21}y_{t2} + \cdots + \gamma_{m1}y_{tm} + \beta_{11}x_{t1} + \cdots + \beta_{k1}x_{tk} &= \varepsilon_{t1} \\
 \gamma_{12}y_{t1} + \gamma_{22}y_{t2} + \cdots + \gamma_{m2}y_{tm} + \beta_{12}x_{t1} + \cdots + \beta_{k2}x_{tk} &= \varepsilon_{t2} \\
 &\vdots \\
 \gamma_{1m}y_{t1} + \gamma_{2m}y_{t2} + \cdots + \gamma_{mm}y_{tm} + \beta_{1m}x_{t1} + \cdots + \beta_{km}x_{tk} &= \varepsilon_{tm}
 \end{aligned}$$

# Simultaneous Equations Model

Structural Form model:

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix}_t \begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1m} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{m1} & \gamma_{m2} & \cdots & \gamma_{mm} \end{bmatrix} + \\
 \begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix}_t \begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1m} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m1} & \beta_{m2} & \cdots & \beta_{mm} \end{bmatrix} = \begin{bmatrix} \varepsilon_{1t} & \varepsilon_{2t} & \cdots & \varepsilon_{mt} \end{bmatrix}$$

or  $y'_t \Gamma + x'_t B = \varepsilon'_t$

# Simultaneous Equations Model

Two endogenous variables models:

$$y_{1t} = \beta_{10} + \beta_{12}y_{2t} + \gamma_{11}x_{1t} + \varepsilon_{1t}$$

$$y_{2t} = \beta_{20} + \beta_{21}y_{1t} + \gamma_{21}x_{1t} + \varepsilon_{2t}$$

## Example

### Keynesian Income Determination Model

Consumption Function  $C_t = \beta_0 + \beta_1 Y_t + \varepsilon_{1t}$

Income Identity  $Y_t = C_t + I_t$

# System Equations Model

Simultaneous Bias occurs when  $E(y_t \varepsilon_{1t}) \neq 0$

OLS estimators will be biased, inconsistent, and inefficient.

$$E(\hat{\beta}) \neq \beta \quad \text{Biased}$$

$$p \lim(\hat{\beta}) \neq \beta \quad \text{Inconsistent}$$

# System Equations Model

**Structural Form Model**  
shows structural relationship between endogenous variables and exogenous variables & lagged endogenous variables.

$$C_t = \beta_0 + \beta_1 Y_t + \varepsilon_{1t}$$

$$Y_t = C_t + I_t$$

**Reduced Form Model**  
shows relationship only between endogenous variables and exogenous variables.

$$C_t = \pi_0 + \pi_1 I_t + w_{1t}$$

$$Y_t = \pi_2 + \pi_3 I_t + w_{2t}$$

where

$$\pi_0 = \frac{\beta_0}{1 - \beta_1} \quad \pi_1 = \frac{\beta_1}{1 - \beta_1}$$

$$\pi_2 = \frac{\beta_0}{1 - \beta_1} \quad \pi_3 = \frac{1}{1 - \beta_1}$$

$$w_{1t} = \frac{\varepsilon_{1t}}{1 - \beta_1} \quad w_{2t} = \frac{\varepsilon_{1t}}{1 - \beta_1}$$

# System Equations Model

## Identification problem:

- Over-identified system
- Exactly-identified system
- Under-identified system

# Single Equation Estimation Methods

## Indirect Least Squares (ILS)

Only for exactly-identified system

- Estimate Reduced form model using OLS
- Solve for estimated  $\beta$

Structural Form Model:

$$Y\Gamma + XB = E$$

Reduced Form Model:

$$Y = X\Pi + V$$

- ILS: 1. Estimate Reduced Form  $\hat{\Pi} = (X'X)^{-1} X'Y$
2. Solve for B  $\hat{B}_{ILS} = -\hat{\Pi}\Gamma$

# Single Equation Estimation Methods

## Two-Stage Least Squares (2SLS)

For both Exactly and Over-identified systems

1. Estimate Reduced form model using OLS

Predict endogenous variable ( $\hat{Y}$ )

2. Use  $\hat{Y}$  as Instrumental Variables (IV) for lagged endogenous variables instead of using actual  $Y$ , then, obtain matrix  $\hat{Z}$ , then estimate model using IV technique

$$\hat{B} = (\hat{Z}'\hat{Z})^{-1} \hat{Z}'Y$$

# System Equations Estimation Methods

## Three-Stage Least Squares (3SLS)

If there exists relationship of the error terms across equations, system equation estimation will lead to more asymptotically efficient estimators.

2SLS + construct estimated  $\Sigma$  matrix

3-stage perform FGLS using estimated  $\Sigma$  matrix

$$\hat{B} = \left[ \hat{Z}' (\hat{\Sigma}^{-1}) \hat{Z} \right]^{-1} \hat{Z}' (\hat{\Sigma}^{-1}) y$$

# OLS, ILS, 2SLS, 3SLS

## Single Equation Estimation Methods

OLS: **biased, inconsistent, & inefficient**

ILS: **biased, consistent, & Asymptotically efficient**

2SLS: **biased, consistent, & Asymptotically efficient**

## System Equation Estimation Methods

3SLS: **biased, consistent?, & More Asymptotically efficient**

- but if there exists **specification error** in one or any equation(s), the **specification error problem** will spread through all equations in the system  
→ **Inconsistent.**

# Specification Test – Hausman Test

Hypothesis OLS estimators are consistent compared to alternative method estimators.

Reject  $H_0$  means there is exists endogeneity (simultaneous) bias (or  $x$ 's are endogenous).

Fail to Reject  $H_0$  means there is no endogeneity (simultaneous) bias (or  $x$ 's are exogenous).