According to the calculation above, the result can be summarized into Table 2.

Table 2 shows joint probability of random variable X and Y

		X			
		-1	0	1	
	1	0.11	0.08	0.05	$f(Y=1) =$ 0.24
Y	2	0.09	0.05	0.03	$f(Y=2) =$ 0.17
	3	0.35	0.07	0.17	$f(Y=3) =$ 0.59
		$f(X=-1) =$ 0.55	$f(X=0) =$ 0.2	$f(X=1) =$ 0.25	$f(X) =$ 1 $f(Y) =$ 1

1.2.5 Conditional Probability Density Function

Conditional probability density function is the probability of one event given that some events have already occurred. The function is written as,

$$f(X|Y) = P(X = x|Y = y)$$

This function can be obtained from the joint probability density function through,

$$f(X|Y) = \frac{f(X,Y)}{f(Y)}$$

Example: According to Table 2, find $f(X = 1|Y = 2)$ and $f(Y = 2|X = 0)$

$$f(X=1|Y=2) = \frac{f(X=1, Y=2)}{f(Y=2)} = \frac{0.03}{0.17}$$

$$f(Y=2|X=0) = \frac{f(Y=2, X=0)}{f(X=0)} \quad \frac{0.05}{0.20}$$

Class practice ☺

$$
\begin{aligned}
(X = 0|Y = 1) &= \\
&= \\
&= \\
&= \\
(Y = 2|X = 0) &= \\
&= \\
&=
\end{aligned}
$$

$f(Y=1) =$

$f(X=0) =$

$f(X=0, Y=1) =$

$f(Y=2, X=0) =$

$$f(X|Y) = \frac{f(X,Y)}{f(Y)}$$

$$f(Y|X) = \frac{f(X,Y)}{f(X)}$$

Example: Let event A be tossing the dice once and the point is odd number and B be the tossing the dice once and the point is at least 5. Find the probability that the point coming up is odd given that the point has to be at least 5.

Answer A and B will occur simultaneously if the point from tossing the dice is 5; so, the joint probability of A and B is $\frac{1}{6}$. The probability that B occurs is $\frac{2}{6}$. Hence, the conditional probability of A given B is

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{\frac{1}{6}}{\frac{2}{6}} = \frac{1}{2}\#$$

1.2.6 Statistical Independence

Two random variables are **independent** if the resulting value of one variable does not affect the resulting value of the other; namely,

$$f(X,Y) = f(X)f(Y)$$

Example: Consider Mr. Ake's expenditure for a meal and the Miss Somsri's expenditure for a dessert. Given that they do not know each other, the realization of Mr. Ake's expenditure does not imply the realization of Miss Somsri's expenditure. We can, thus, conclude that the expenditures of these two people are independent#

Example: Consider drawing cards sequentially from the standard 52-card deck without putting it back into the deck. Once the first card is drawn, the probability of drawing the second card will be influenced because the amount of cards in the deck is reduced. In this case, it can be concluded that drawing the first and second card are not independent#

1.3 Expectation, Variance, Covariance and Correlation

1.3.1 Mean or Expected Value

Because the value of random variable hinges on the value of random results of experiment which cannot be determined certainly, statisticians have invented the measures of central tendency of the random variable. One of them is **expected value**, indicating the mean of the random variable.

For discrete random variable, the expected value is calculated by;

$$E(X) = \sum_{i=i}^{n} x_i f(x_i) = x_1 f(x_1) + x_2 f(x_2) + \ldots + x_n f(x_n)$$

Statistical Independence

$$f(x, y) = f(x)\, f(y)$$

E.g. A bag contains 3 balls numbered 1, 2, and 3. Two balls are drawn at random, with replacement, from the bag.

Let X denote the number of the 1st ball drawn and Y the number of the 2nd ball drawn.

		X		
		$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
Y		1	2	3
$\frac{1}{3}$	1	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
$\frac{1}{3}$	2	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
$\frac{1}{3}$	3	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

$$f(X=1, Y=1) = \frac{1}{9}$$

$$f(1,1) = \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)$$

Continuous Joint PDF

The PDF $f(x,y)$ of two continuous variables X and Y is such that

① $f(x,y) \geq 0$

② $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

③ $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = P(a \leq x \leq b, c \leq y \leq d)$

Marginal PDF (continuous)

$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$ Marginal PDF of X

$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$ Marginal PDF of Y e.g. $f(x,y) = 2-x-y$

$0 \leq x \leq 1 ; 0 \leq y \leq 1$

$$\int_0^1\int_0^1 2-x-y \, dx\, dy = 1$$

$$\int_0^1 \left[2x - \frac{x^2}{2} - xy \Big|_0^1 \right] dy$$

$$\int_0^1 \left[\frac{3}{2} - y \right] dy$$

$$\frac{3y}{2} - \frac{y^2}{2} \Big|_0^1 = 1$$

Marginal PDF of X

$$f(x) = \int_0^1 2 - x - y \, dy$$

$$= \left[2y - xy - \frac{y^2}{2} \right] \Big|_0^1$$

$$= \frac{3}{2} - x$$

Marginal PDF of Y

$$f(y) = \int_0^1 2 - x - y \, dx$$

$$= 2x - \frac{x^2}{2} - xy \Big|_0^1$$

$$= \frac{3}{2} - Y$$

For continuous random variable, the expected value is calculated by,

$$E(X) = \int_a^b x f(x) dx$$

where;

$E(X)$ is the measure of central tendency of random variable, resulting from repeated trial of experiment.

$\sum_{i=i}^{n} x_i f(x_i)$ is the average of random variable weighted by the probability corresponding to each value.

a and b are the lowest and highest constant possible respectively.

Example: Find the expected value of rolling two dice once (Figure 1)

$$E(X) = \sum_x x f(x)$$

$$= 2\left(\frac{1}{36}\right) + 3\left(\frac{2}{36}\right) + \dots + 12\left(\frac{1}{36}\right)$$

$$= 7$$

x	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Crucial properties of expected value include:

1. $E(b) = b$

2. $E(aX + b) = aE(X) + b$

3. $E(XY) = E(X)E(Y)$; given that X and Y are independent

4. $E(g(X)) = \sum_x g(X)f(X)$

where a and b are constant.

Conditional expectation value is the expectation value of random variable under some conditions such as expected value of X conditional on Y or $E(X|Y = 5)$

Let $f(X,Y)$ be the joint probability function of X and Y. The expectation of X conditional on some value of Y is defined as,

For discrete random variable $E(X|Y = y) \;=\; \sum_X X_i f(X|Y = y)$

For continuous random variable $E(X|Y = y) \;=\; \int_{-\infty}^{\infty} X_i f(X|Y = y)$

Example

② $E(aX+b) = aE(X)+b$

$$E(aX+b) = \sum (ax_i+b) f(x_i)$$

$$= \sum ax_i f(x_i) + \sum b f(x_i)$$

$$= a\underbrace{\sum x_i f(x_i)}_{E(X)} + b\underbrace{\sum f(x_i)}_{1}$$

$$= aE(X)+b$$

③ If X and Y are independent RV then

$$E(XY) = E(X)E(Y)$$

Joint PDF

$$E(XY) = \sum\sum x_i y_j f(x_i y_j)$$

$$f(x_i y_j) = f(x_i) f(y_j)$$ Statistical Independence

$$E(XY) = \sum x_i f(x_i) \sum y_j f(y_j)$$

$$= E(X)E(Y)$$

However,

$$E\left(\frac{X}{Y}\right) \neq \frac{E(X)}{E(Y)}$$

even if X and Y are independent.

4. If X is a R.V. with PDF $f(x)$ and if $g(x)$ is any function of X, then

$$E(g(X)) = \sum_x g(x)f(x) \text{ if } X \text{ is discrete}$$

$$= \int_{-\infty}^{\infty} g(x)f(x)\,dx \text{ if } X \text{ is continuous.}$$

X	-2	1	2
$f(x)$	$\frac{5}{8}$	$\frac{1}{8}$	$\frac{2}{8}$

$$E(X) = -2\left(\frac{5}{8}\right) + 1\left(\frac{1}{8}\right) + 2\left(\frac{2}{8}\right)$$

$$E(X) = \sum_x g(x) f(x)$$

if $g(x) = X^2$

$$E(X^2) = 4\left(\frac{5}{8}\right) + 1\left(\frac{1}{8}\right) + 4\left(\frac{2}{8}\right) = \frac{29}{8}$$

$$E(X)^2 \neq E(X^2)$$

Practice for fun

Y \ X	-1	0	2	4
2	0.08	0.27	0	0.16
5	0	0.10	0.04	0.35

$E(X) = \sum_x x f(x)$

$E(Y) = \sum_y y f(y)$

if $g(x) = x^2$

$E(x^2) =$

Conditional Expectation

$$E(X|Y=y) = \sum_x x\, f(x|Y=y) \text{ if } x \text{ is discrete}$$

$$f(x|Y=y) = \frac{f(X=x, Y=y)}{f(y)}$$

Example

		X			
		-2	0	2	3
Y	3	0.27	0.08	0.16	0
	6	0	0.04	0.10	0.35

$$E(Y|X=2) = \sum_y y\, f(y|X=2)$$

$$f(y|X=x) = \frac{f(X=x, Y=y)}{f(x)}$$

$$= 3\left(\frac{0.16}{0.26}\right) + 6\left(\frac{0.10}{0.26}\right) = 4.15$$

$f(Y=3|X=2)$ $f(Y=6|X=2)$

$$f(Y=3 \mid X=2) = \frac{f(Y=3, X=2)}{f(X=2)}$$

$$= \frac{0.16}{0.26}$$

$$f(Y=6 \mid X=2) = \frac{f(Y=6, X=2)}{f(X=2)}$$

$$= \frac{0.10}{0.26}$$