Chapter 2

Foundations of Probability Conditional Probability and Independence

Sections 2.3 ,2.4, and 2.5

- A Review of Set Notation
- A Probabilistic Model for an Experiment : The Discrete Case
- Calculating the Probability of an Event: The Sample-Point Method

- A probability experiment is any action for which an outcome, response, or measurement is obtained that cannot be predicted with certainty.

- A sample space is the set of all possible simple outcomes, responses, or measurements of an experiment.

- An event in a discrete sample space S is a collection of sample points – that is, any subset of S.

  (An event is a subset or collection of outcomes from the sample space.)

- A discrete sample space is one that contains either a finite or a countable number of distinct sample points.

- A simple event is one that cannot be broken down any further; that is, it is an individual outcome from the sample space.

Example : Suppose two dice are tossed and the numbers on the upper faces are observed.

(i) List the sample points for this experiment.

(ii) Define the following subsets of S.

A: The sum of the two numbers is greater than nine.

B: The two numbers are the same.

Solutions :

(i) The simple events may be denoted by

E_1=(1,1)	E_7=(2,1)	E_{13}=(3,1)	E_{19}=(4,1)	E_{25}=(5,1)	E_{31}=(6,1)
E_2=(1,2)	E_8=(2,2)	E_{14}=(3,2)	E_{20}=(4,2)	E_{26} =(5,2)	E_{32}=(6,2)
E_3=(1,3)	E_9=(2,3)	E_{15}=(3,3)	E_{21}=(4,3)	E_{27} =(5,3)	E_{33}=(6,3)
E_4=(1,4)	E_{10}=(2,4)	E_{16}=(3,4)	E_{22}=(4,4)	E_{28} =(5,4)	E_{34}=(6,4)
E_5=(1,5)	E_{11}=(2,5)	E_{17}=(3,5)	E_{23}=(4,5)	E_{29} =(5,5)	E_{35}=(6,5)
E_6=(1,6)	E_{12}=(2,6)	E_{18}=(3,6)	E_{24}=(4,6)	E_{30}=(5,6)	E_{36}=(6,6)

Thus there are 36 sample points in S, and S = $\{ E_1, E_2, \ldots, E_{36} \}$.

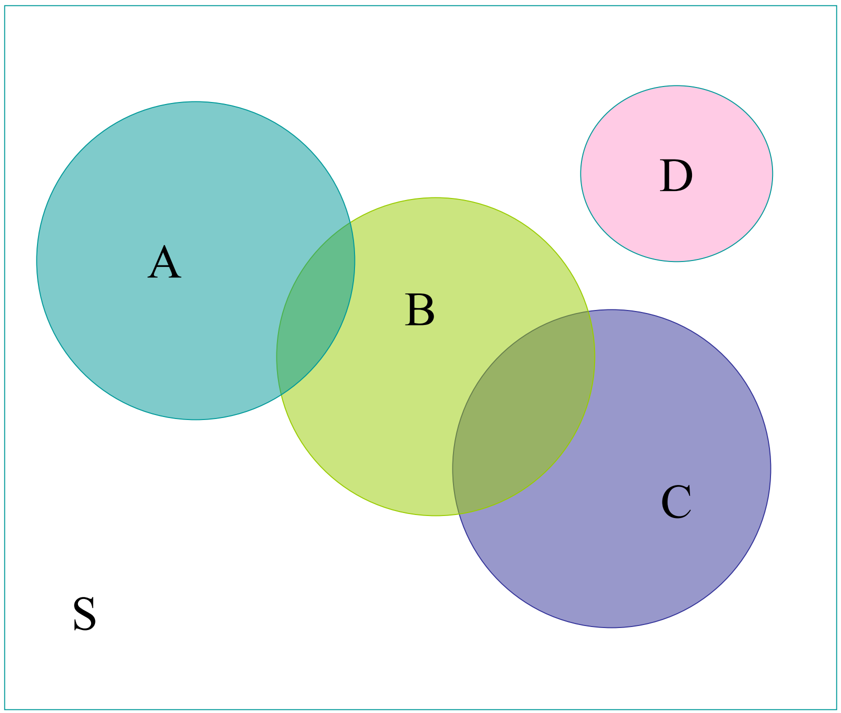
(ii) Event A = $\{ E_{24}, E_{29}, E_{30}, E_{34}, E_{35}, E_{36} \}$.

Event B = $\{E_1, E_8, E_{15}, E_{22}, E_{29}, E_{36} \}$.

To facilitate the theory we need to establish a set theory foundation.

- Sets – denoted by capital letters A,B,C,…
- Element of sets will be denoted by small letters.
- A is a subset of B , $(A \subset B)$, if every point in A is also in B.
- The empty or null set, $\varnothing$, is the set consisting of no points.

Venn Diagrams



- $A \cup B$ "union" - set of all elements that are in A, in B , or in both.
- $A \cap B$ "intersection" - set of all elements in both A,B.
- $\overline{A}$ "complement" - set of all elements not in A.
- A, B are disjoint or mutually exclusive if

$$A \cap B = \varnothing$$

- Commutative laws: $$A \cup B = B \cup A$$ $$A \cap B = B \cap A$$

- Associative laws: $$(A \cup B) \cup C = A \cup (B \cup C)$$ $$(A \cap B) \cap C = A \cap (B \cap C)$$

- Distributive laws: $$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$ $$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

- De Morgan's laws: $$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$$ $$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$$

We will use set theory and set operations to aid probability calculations.

Roughly speaking, the probability of an event is a numerical value that “ measures” the “ chances” that the event will “occur”.

We say an event occurs if it contains the realized outcome of the experiment.

Example : From previous example,

- If the outcome was (6,6)
  - then event A occurred.

- If the outcome was (1,5)
  - then event A did not occur.

Definition : The probability of an event A, denoted by P(A), is a numerical value that represents the proportion of time the event is expected to occur under repeated trials of the experiment.

Suppose every outcome is equally likely.

N = number of possible outcomes in S

n = number of outcomes in A

Then

$$P(A) = \frac{n}{N}$$

Axiom of Probability

Suppose S is a sample space associated with an experiment. To every event A in S (A is a subset of S), the probability of A (P(A)) satisfies :

1. $0 \le P(A) \le 1$
2. $P(S) = 1$

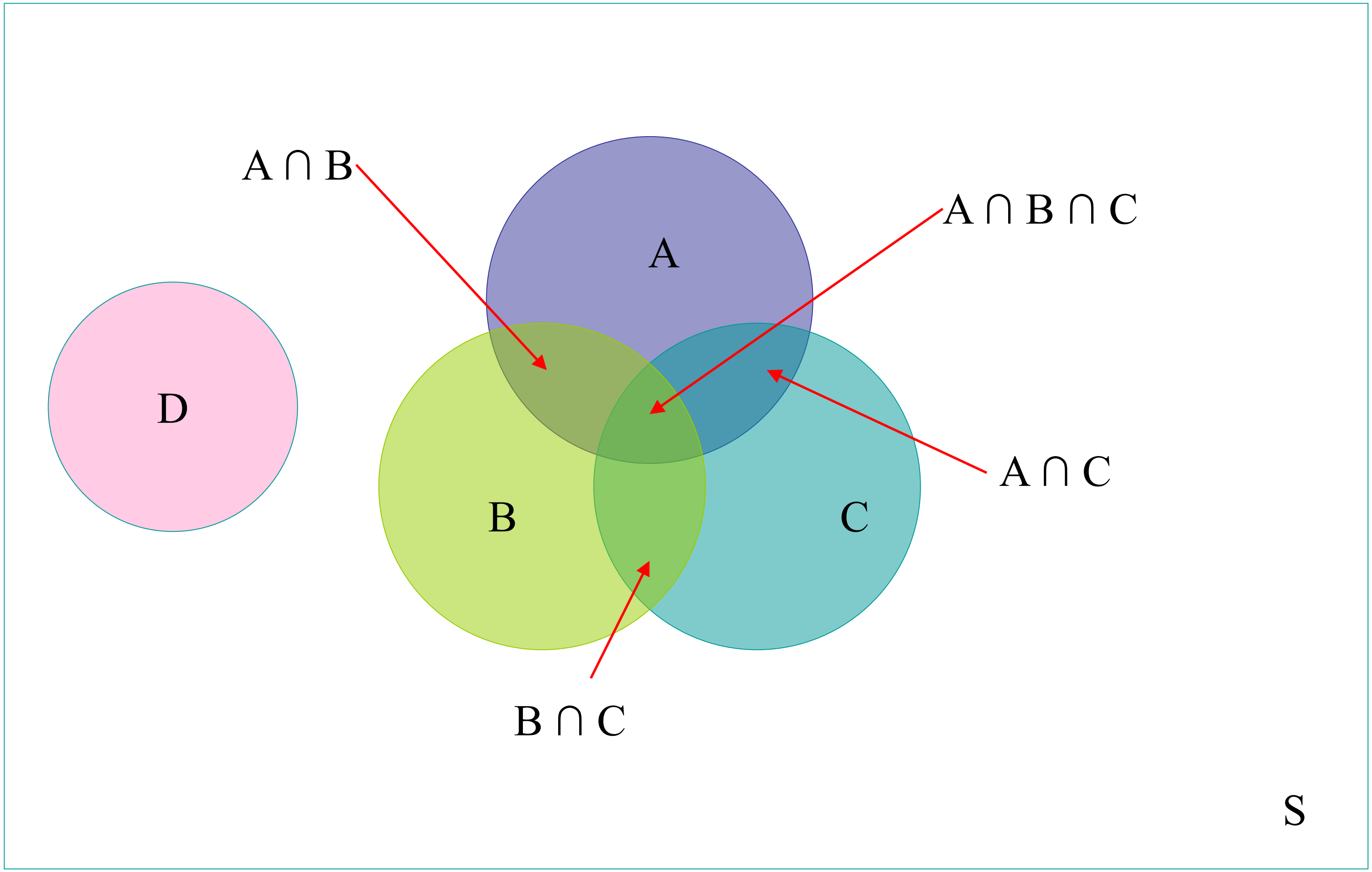
   $P(\varnothing) = 0$
3. If $A_1, A_2, \ldots$ form a sequence of pairwise mutually exclusive events in S (that is, $A_i \cap A_j = \varnothing$ if $i \ne j$), then

$$P(A_1 \cup A_2 \cup \ldots) = \sum_{i=1}^{\infty} P(A_i)$$

Useful Facts

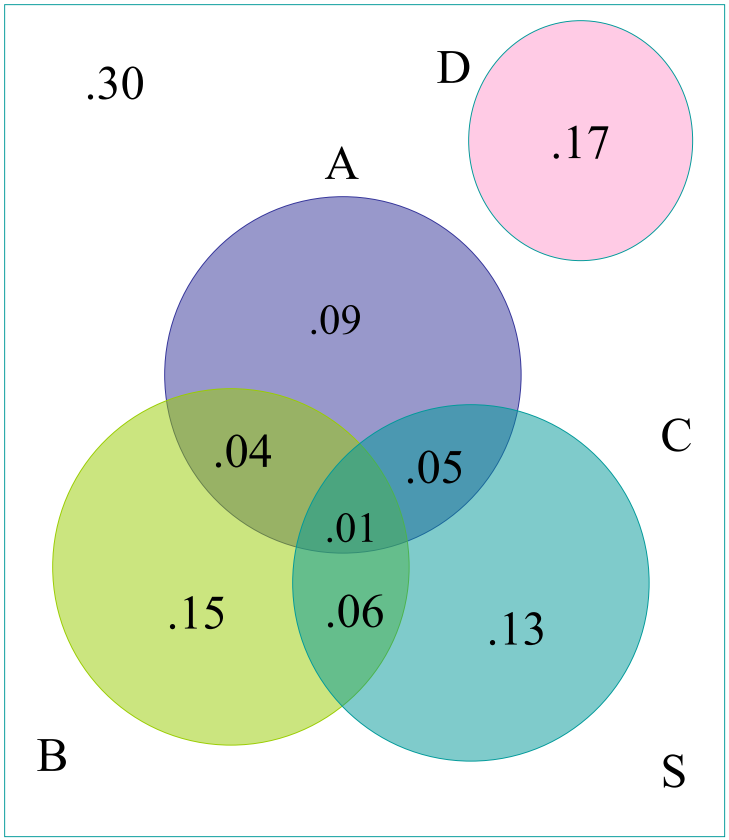
1. $P(\overline{A}) = 1 - P(A)$
2. $P(A \cap B) = 0 \quad \Leftrightarrow \quad \text{A and B are disjoint}$
3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
4. $$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Venn Diagram



Examples

Example 1 : Given



1. $P(A \cup B)$
2. $P(\overline{A})$
3. $P(A \cup B \cup C)$
4. $P(A \cap D) =$
5. $P(\overline{A} \cap \overline{B})$
6. $P(\overline{A} \cap \overline{B} \cap \overline{C})$
7. $P(\overline{A} \cap \overline{B} \cup C)$

Example 2 :

Given $P(A) = .6, P(B) = .5, P(A \cup B) = .8$

Determine

(1) $P(A \cap B)$

(2) $P(A \cap \overline{B})$

(3) $P(A \cap \overline{B}) + P(B \cap \overline{A})$

Example 3 : 3 glasses of cola – C, D, P

- drink all 3 glasses in succession
- and rank them for taste

- List the sample space
- P(D ranked first)
- P(D ranked first and P ranked last)

Solutions

- S = { CDP, CPD, DCP, DPC, PCD, PDC }
  There are 6 outcomes in S.

- P(D ranked first) =

- P(D ranked first and P ranked last) =

Example 4 : A vehicle arriving at an intersection can turn right, turn left, or continue straight ahead. The experiment consists of observing the movement of a single vehicle through the intersection.

(a) List the sample space for this experiment.

(b) Assuming that all sample points are equally likely, find the probability that the vehicle turns.

Solutions

(a) Let L = vehicle turns left

R = vehicle turns right

S = vehicle continues straight

Then the sample space would be

(b) P(vehicle turns)

Example 5 : A boxcar contains six complex electric systems. Two of the six are to be randomly selected for testing and then classified as defective or not defective.

(a) If two of the six systems are actually defective, find the probability that at least one of the two systems tested will be defective. Find the probability that both are defective.

(b) If four of the six systems are actually defective, find the probabilities indicated in (a)

Section 2.6

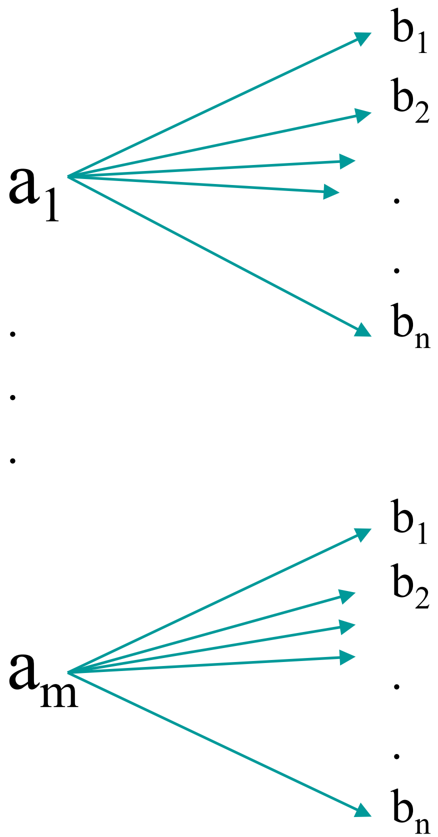
Tools for Counting Sample Points

Recall : If a sample space contains N equiprobable sample points and an event A contains exactly n_a sample points, it is easily seen that $P(A) = n_a / N$.

- Multiplication Rule (The *mn* rule) :

With m elements $a_1, a_2, \ldots, a_m$ and n elements $b_1, b_2, \ldots, b_n$ it is possible to form $mn = m \times n$ pairs containing one element from each group.

Tree Diagram



Example : Toss a coin twice. Find the number of sample points in S, the sample space for the experiment, and find the probability that the outcome for the 2^{nd} time is the same as the 1^{st} time.

Solutions :

- **Permutations:** An ordered arrangement of *r* distinct objects is called a ***permutation***. The number of ways of ordering *n* distinct objects taken *r* at a time will be designated by the symbol P_r^n .

P_r^n = number of different permutations of *n* objects taken *r* at a time

$$= n(n-1)(n-2)\ldots(n-r-1) = \frac{n!}{(n-r)!}$$

In particular,

$$P_n^n = n!$$

- **Combinations:** The number of combinations of *n* objects taken *r* at a time is the number of subsets, each of size *r*, that can be formed from the n objects. This number will be denoted by C_r^n or $\binom{n}{r}$.

$$C_r^n = \binom{n}{r} = \text{ number of combinations of } n \text{ objects taken } r \text{ at a time}$$

$$= \frac{n!}{r!(n-r)!}$$

Useful Formula :

$$\binom{n}{r} = \binom{n}{n-r}$$

$$\frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!(n-(n-r))!}$$

$$= \frac{n!}{r!(n-r)!}$$

Partitions: The number of ways of partitioning n distinct objects into k distinct groups containing $n_1, n_2, \ldots, n_k$ objects, respectively, where each object appears in exactly one group and $n_1 + n_1 + \ldots + n_k = n$, is

$$N = \binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! n_2! \ldots n_k!}$$

multinomial coefficients

$$(y_1 + y_2 + \ldots + y_k)^n = \sum \binom{n}{n_1, n_2, \ldots, n_k} y_1^{n_1} \ldots y_k^{n_k}$$

From a box that contains n distinct objects, randomly draw out k objects.

Number of Elements in Sample Space

	Ordered	Unordered
Without replacement	P_k^n	$\binom{n}{k}$
With replacement	n^k	$\binom{k+(n-1)}{k}$

Equally likely outcomes (P_k^n, n^k, $\binom{n}{k}$)

Outcome not equally likely ($\binom{k+(n-1)}{k}$)

Examples

Example 1 : There are 10 homes, selecting 3 of them.

(a) Ordered arrangement

(b) Unordered

(c) If there are 4 new homes and 6 old homes. Find the probability that all 3 selected are new.

Example 2: How many 4 digits number can be formed from digits 1, 3, 5, 7, and 9.

(a) If each digit can be used only once.

____ ____ ____ ____

(b) If each digit can be used repeatedly.

____ ____ ____ ____

(c) If each digit can be used only once and none are digit 7.

____ ____ ____ ____

Example 3 : 5 – types of receivers (A,B,C,D,E)

4 – CD players (A,B,C,D)

3 – speakers (A,B,C)

4 – cassette decks (A,B,C,D)

(a) Find the total number of ways to select 4 pieces, one each type, for the home theater system.

(b) Both CD player and receiver are from brand D.

(c) None are D.

(d) At least one D.

(e) P(at least one D)

(f) P(exactly one D)

Example 4 : 20 buses , and cracks have appeared in 8 of them.

(a) How many ways to select 5 from 20?

(b) How many ways to select 5 such that exactly 4 have cracks?

(c) P (exactly 4 of 5 have cracks)

(d) P (at least 4 of 5 have cracks)

Example 5 : Production facility employs :

20 – on day shift

15 – on swing shift

10 – on graveyard shift

Select 6 of them at random.

(a) How many ways can 6 be selected such that all 6 are from day shift?

(b) P (6 from day shift)

(c) P(all 6 workers are from same shift)

(d) P (at least 2 different shifts are represented)

(e) P (at least one shift is unrepresented)

Let A_1 = day shift unrepresented

= only swing shift and graveyard shift represented.

Similarly for A_2, and A_3. Then (e) $= P(A_1 \cup A_2 \cup A_3)$

Section 2.7

Conditional Probability and the Independence of Events

Conditional Probability

In general the probability of an event may (or may not be affected by the knowledge of an occurrence of some other event.

E.g..

(1) Flip a coin twice – observed heads on the 1st toss. Does this knowledge affect the probability of heads on the 2nd toss?

(2) Draw two cards from a deck of 52 in succession without replacement. Does the knowledge of the 1st card affect the probability of the 2nd card being a 5 ?

- Roughly speaking the knowledge of a 5 on the first draw has changed the sample space.
- We refer to this as a conditional probability.

Definition : The conditional probability of an event A given that an event B has occurred is denoted by P (A | B) and is equal to

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Examples

<u>Example 1</u> : Two cards are selected from a standard deck (52 cards) without replacement. What is the probability that both card are King?

Let A = { the 1st card is a King }

B = { the 2nd card is a King }

There are 4 Kings. P(A) = 4/52.

$$P(B \mid A) = \frac{P(B\ and\ A)}{P(A)}$$

$$P(A\ and\ B) = P(B \mid A)P(A) \quad = \frac{3}{51} \cdot \frac{4}{52} = .0045$$

Example 2 : Contingency Table

	Science (S)	Engineering (E)	Arts (A)	Total
Male (M)	70	50	80	200
Female (F)	35	25	40	100
Total	105	75	120	300

If one student is selected. Find the probability that this student

(1) is a female

(2) is an Engineering student, given that this student is female.

Independence of Events

Two events, A and B, are independent if the occurrence or nonoccurrence of A does not change the probability that B will occur (and vice-versa), that is, $P(B \mid A) = P(B)$.

Definition: Event A and B are said to be independent if

$$P(A \mid B) = P(A) \quad \text{or} \quad P(B \mid A) = P(B)$$

Both imply

$$P(A \cap B) = P(A)P(B)$$

Example

	Science (S)	Engineering (E)	Art (A)	Total
Male (M)	70	50	80	200
Female (F)	35	25	40	100
Total	105	75	120	300

Are the events E and F independent?

P(E | F) = .25 P(E) = 75/300 = .25 YES !

P(F | E) = (25/300)/(75/300) = .33

P(F) = 100/300 = .33 YES !!

Section 2.8

Two Laws of Probability

- <u>The Multiplicative Law of Probability</u>

  The probability of the intersection of two events A and B is

$$P(A \cap B) = P(A)P(B \mid A)$$

$$= P(B)P(A \mid B)$$

  If A and B are independent, then

$$P(A \cap B) = P(A)P(B)$$

- <u>The Additive Law of Probability</u>

  The probability of the union of two events A and B is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

  If A and B are mutually exclusive events, $P(A \cap B) = 0$ and

$$P(A \cup B) = P(A) + P(B)$$

- Complement of an Event

The complement of an event A with respect to S, sample space, is the set of all elements in S that are NOT in A, denoted $\overline{A}$

$$P(A) = 1 - P(\overline{A})$$

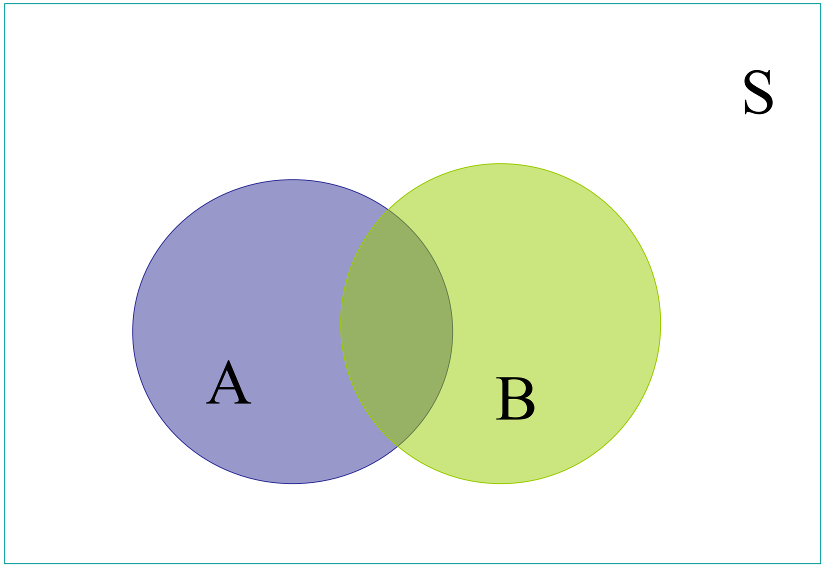
Example

Draw two cards from a deck of 52 in succession without replacement. What is the probability that the 1st and 2nd cards are 5 ?

Solution :

Section 2.9 & Section 2.10

- Calculating the Probability of an Event : The Event – Composition Method
- The Law of Total Probability and Bayes'Rule



Observe the following identity.

(disjoint)

$$A = (A \cap B) \cup (A \cap \overline{B})$$

Thus

$$P(A) = P(A \cap B) + P(A \cap \overline{B})$$

$$= P(A \mid B)P(B) + P(A \mid \overline{B})P(\overline{B})$$

Hence,

$$P(A) = P(A \mid B)P(B) + P(A \mid \overline{B})P(\overline{B})$$

We can apply the same idea to a more general partitions.

Partition of S

Suppose $S = B_1 \cup B_2 \cup \ldots \cup B_k$ and $B_i \cap B_j = \varnothing \,, i \neq j$

(assuming $P(B_i) > 0$)

Then for any event A

$$P(A) = \sum_{i=1}^{k} P(A \mid B_i) P(B_i)$$

We can easily derive a formula known as Bayes' Rule.

Assume $\{B_1, B_2, \ldots, B_k\}$ forms a partition of S and $P(B_i) > 0$ for all i.

Then $$P(B_j \mid A) = \frac{P(A \mid B_j)P(B_j)}{\sum_{i=1}^{k} P(A \mid B_i)P(B_i)}$$ ← $P(A)$

Example

Test for a disease D, and we estimate that 10% of the population has the disease . Test gives a false-positive 5% of the time and true-positive 85%. Determine

(i) P(a person test's positive)

(ii) P(have D | test positive)