Mean-Variance Analysis and the Pricing of Risk

EE431/438

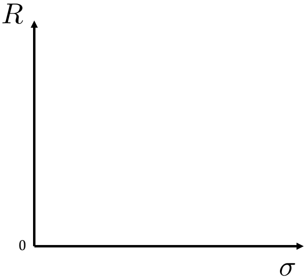
Copeland, Thomas E. and J. Fred Weston, Financial Theory and Corporate Policy (4th ed), Addison-Wesley, 2005: Ch5 (pp 101 -141)

August 2011

Introduction

- Risk averse individuals dislike risk but this doesn't mean they are never prepared to take any risks. It just means they need to be compensated for the risk they take by way of a greater expected return
- State preference framework = general approach (used extensively in economic theories)
- It is difficult (if not possible) to list list all payoffs offered in different state of nature.
- mean-variance analysis (less general, more practical)
- Risks: deviation from expected outcome
- Expected outcome = expected return
- Risks: variance, standard deviation
- Rate of returns $\rightarrow$ asset prices

Introduction (2)



- $E(X) = \sum_{i=1}^{n} p_i X_i$
- $\sigma^2 = E[(x-\mu)^2]$, S.D = $\sigma = \sqrt{E[(x-\mu)^2]}$
- Indifference curve on mean-variance plane
- Risk averse, Risk lovers, Risk neutral

Measuring Return for a Single Asset (1)

p_i=probabilty	End-of-Period Price per share	R_i=Return(%)
0.1	\$20.00	
0.2	\$22.50	
0.4	\$25.00	
0.2	\$30.00	+20
0.1	\$40.00	+60
1	(current price = \$25)	

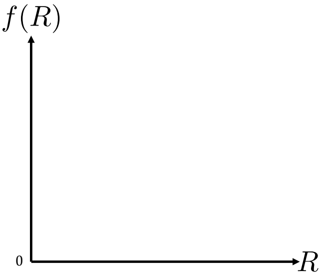
Measuring Return for a Single Asset (2)

- $E(X) = \sum_{i=1}^{n} p_i X_i$
- Expected end-of-period price
  $E(P) = 0.1 \times (......) + 0.2 \times (......) + 0.4 \times (......) + 0.2 \times (......) + 0.1 \times (......) = \$26.50$
- Expected Return
  $$E(R) = \frac{E(P) - P_0}{P_0} = \frac{26.50 - 25}{25} = 6\%$$
  or
  $E(R) = 0.1 \times (......) + 0.2 \times (......) + 0.4 \times (......) + 0.2 \times (......) + 0.1 \times (......) = 6$

Measuring Return for a Single Asset (2)

- property 1: $E(X+a) = E(X) + a$
  - proof : $E(X+a) = \sum_{i=1}^{n} p_i(X_i + a)$
- property 2: $E(aX) = aE(X)$
  - proof : $E(aX) = \sum_{i=1}^{n} p_i(aX_i)$

Measuring Dispersion for a Single Asset



- Dispersion $\rightarrow$ how the returns(outcome) are distributed
- Statistically, there are many measurements (for example, range).
- Here, we focus on variance and stardard deviation.

Variance and Standard Deviation (1)

- $VAR(X) = \sum_{i=1}^{n} p_i (X_i - E(X))^2$
- From the previous example, the variance of the end-of-period price is
  $VAR(X) = \sum_{i=1}^{n} p_i (X_i - E(X))^2 =$
  $0.1(..... - 26.5)^2 + 0.2(...... - 26.5)^2 + 0.4(25 - 26.5)^2 + 0.2(30 - 26.5)^2 + 0.1(40 - 26.5)^2 = 29$
- Standard deviation $\sigma_x = \sqrt{VAR(X)}$
- Therefore, $\sigma = \sqrt{29} = \$5.39$

Variance and Standard Deviation (2)

- Property 1: The variance of a random variable plus a constant is equal to the variance of the random variable.
  $VAR(X + a) = VAR(X)$
  - proof: $VAR(X + a) = E((X + ...) - E(X + a))^2$
  - $E(X + a) = ...$
  - $VAR(X + a) = E(X + ... - (......... + ...))^2 = E(X + ... - .... - ....))^2 = ...$
- Property 2: The variance of a random variable multiplied by a constant is equal to the constant squared times the variance of the random variable.
  $VAR(aX) = a^2 VAR(X)$
  - proof: $VAR(aX) = E((....X) - E(.....X))^2$
  - $E(aX = ...+ VAR(aX) = E(aX - (...................))^2 = E(.......(.......... - ....))^2 =$ ....

Variance and Standard Deviation (3)

- We already have $VAR(P)$ (end-of-period price) = \$ 29
- Rate of return = $\dfrac{P - P_0}{P_0}$; P_0 is the current price, \$ 25
- $VAR(\dfrac{P - P_0}{P_0}) = ?$
- $VAR(P - P_0) = VAR(P + (-P_0)) =$ ....
- $VAR(\dfrac{P - P_0}{P_0}) = VAR(\dfrac{1}{P_0}(P - P_0)) =$ ....
- Therefore, the variance of the rate of return,

  $$VAR(R) = \frac{VAR(P)}{P_0^2} = \frac{\text{..........}}{\text{.........}} = 4.64\%$$

Calculating the Mean and Variance of a Two-asset Portfolio (1)

- a portfolio which a % of wealth invested in asset X and $b = (1 - a)$% in asset Y
- X (Y) is the rate of return of asset X (Y)
- portfolio returns: $R_p =$ ....
- example: wealth = 200, $X = 10\%$, $Y = 20\%$; a=100%, a=50%,
- R_p is random: $E(R_p) = E[aX + bY] =$ ....+ ....
- $$\begin{aligned} E(VAR_p) &= E[(R_p - E(R_p)]^2 \\ &= E[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots]^2 \\ &= E[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots]^2 \\ &= E[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots] \\ &= \end{aligned}$$

Calculating the Mean and Variance of a Two-asset Portfolio (2)

- $VAR_p =$ $a^2 VAR(X) + b^2 VAR(Y) + 2ab[(X - E(X))(Y - E(Y)]$
- $COV(X, Y) = E[(X - E(X))(Y - E(Y))]$
- $VAR_p = a^2 VAR(X) + b^2 VAR(Y) +$ ....
- Covariance is a measure of the way in which two variables move in relation to each other
- If the covariance is positive(negative), the variables move in the same(opposite) direction
- Covariance $\rightarrow$ how a single asset constributes to portfolio risk
- Notice that the covariance with itself is variance.

Calculating the Mean and Variance of a Two-asset Portfolio (3)

- Example

p_i=probabilty	X(%)	Y(%)
0.2	11	-3
0.2	9	15
0.2	25	2
0.2	7	20
0.2	-2	6
1		

- $E(X) = ... \times 11 + ... \times 9 + 0.2 \times 25 + 0.2 \times 7 + 0.2 \times (-2) = 10\%$
- $E(Y) = ... \times (-3) + ... \times 15 + 0.2 \times 2 + 0.2 \times 20 + 0.2 \times (6) = 8\%$

Calculating the Mean and Variance of a Two-asset Portfolio (4)

- $E(X) = 10\%$, $E(X) = 8\%$
- $VAR(X) = \sum_{i=1}^{n} p_i(X_i - E(X))^2$
- $VAR(X) = ... \times (11 - ....)^2 + ... \times (9 - ....)^2 + 0.2 \times (25 - 10)^2 + 0.2 \times (7 - 10)^2 + 0.2 \times ((-2) - 10)^2 = 0.0076(\sigma = \sqrt{VAR(X)} = 0.0872 = 8.72\%)$
- $VAR(Y) = ... \times ((-3) - ....)^2 + ... \times (15 - ....)^2 + 0.2 \times (2 - 8)^2 + 0.2 \times (20 - 8)^2 + 0.2 \times (6 - 8)^2 = 0.00708(\sigma = \sqrt{VAR(Y)} = 0.0841 = 8.41\%)$
- $COV(X, Y) = \sum_{i=1}^{n} p_i(X_i - E(X))(Y_i - E(Y))$
- $COV(X, Y) = ... \times (11 - ....)((-3) - ....) + ... \times (9 - ....)(15 - ....) + 0.2 \times (25 - 10)(2 - 8) + 0.2 \times (7 - 10)(20 - 8) + 0.2 \times ((-2) - 10)(6 - 8) = -0.0024.$

Calculating the Mean and Variance of a Two-asset Portfolio (5)

- Negative covariance implies that when X produces a return above its average return, Y do the opposite thing
- When we are losing with asset X, we are making profit with asset Y.
- Therefore, the risk is reduced if we invest in both X and Y.
- This is called "the effect of diversification" ; reducing risk by investing in a variety of assets
- Example: $a = 50\%$,$b = 1 - a =$ .... , $E(X) = 10\%$, $E(Y) = 8\%$, $VAR(X) = 0.0076$, $VAR(Y) = 0.00708$, $COV(X, Y) = -0.0024$
  - Formula : $E(R_p) =$ .... + ... and $VAR_p = a^2 VAR(X) + b^2 VAR(Y) +$ ....
  - $E(R_p) =$ .... + ... $= 9\%$
  - $VAR_p =$ .... + ... + ... $= 0.00247$ ($\sigma = \sqrt{VAR_p} = 0.0497 = 4.97\%$

Calculating the Mean and Variance of a Two-asset Portfolio (5)

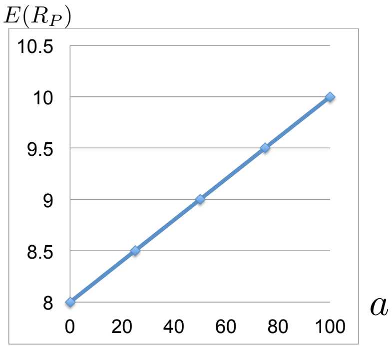
- $E(X) = 10\%$, $E(Y) = 8\%$, $VAR(X) = 0.0076$, $VAR(Y) = 0.00708$, $COV(X, Y) = -0.0024$
- $E(R_p, a = 50\%) = 9\%$, $VAR_{p(a=50\%)} = 0.00247$ $(\sigma = \sqrt{VAR_p} = 0.0497 = 4.97\%$
- The expected return is half way between that offerred by X and by Y
- Risk is considerably less than half way between $VAR(X)$ and $VAR(Y)$

Calculating the Mean and Variance of a Two-asset Portfolio (6)

Percentage in X	Percentage in Y	$E(R_p)(\%)$	$\sigma(R_p)(\%)$
100	0	10	8.72
75		9.5	6.18
50		9.0	4.97
25		8.5	5.96
0	100	8.0	8.41

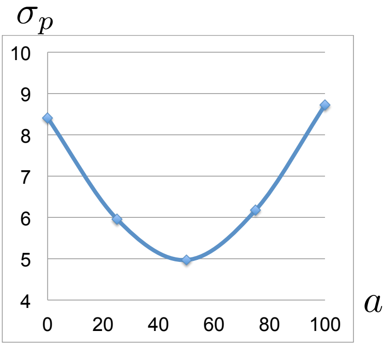
Calculating the Mean and Variance of a Two-asset Portfolio (7)

- The relation between the weight of X in the portfolio and the portfolio return



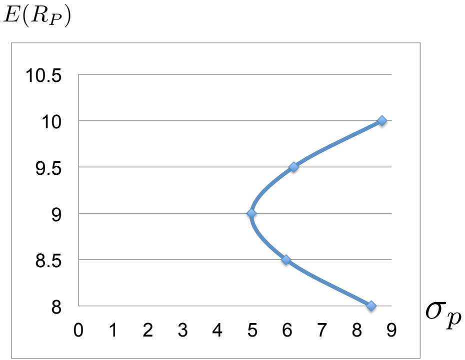
Calculating the Mean and Variance of a Two-asset Portfolio (8)

- The relation between the weight of X in the portfolio standard deviation



Calculating the Mean and Variance of a Two-asset Portfolio (9)

- The relation between the portfolio return and the portfolio standard deviation



Correlation Coefficient (10)

- Theoretically, the possible range of covariance extends all the way from minus infinity to plus infinity.
- We can bound it by dividing it by the product of standard deviation for the two assets
- $r_{x,y} = \dfrac{COV(X,Y)}{\sigma_x \sigma_y}$; $-1 \leq r_{x,y} \leq 1$
- $COV(X,Y) = r_{x,y}\sigma_x\sigma_y$
- from the example; $r_{x,y} = \dfrac{-0.024}{(0.0872)(0.0841)} = -0.33$

The Minimum Variance Portfolio (11)

- How to find the minimum variance portfolio?
- $VAR_p = a^2 VAR(X) + b^2 VAR(Y) + 2abCOV(X, Y)$
- $COV(X, Y) = r_{x,y}\sigma_x\sigma_y$
- $$\begin{aligned} VAR_p &= a^2\sigma_x^2 + (1-a)^2\sigma_y^2 + \ldots\ldots\ldots\ldots\ldots\ldots \\ &= a^2\sigma_x^2 + (\ldots\ldots\ldots\ldots)\sigma_y^2 + (\ldots\ldots\ldots\ldots\ldots\ldots)r_{x,y}\sigma_x\sigma_y \end{aligned}$$
- $$\begin{aligned} VAR_p &= \ldots\ldots\sigma_x^2 + (\ldots\ldots\ldots\ldots)\sigma_y^2 + (\ldots\ldots\ldots\ldots\ldots\ldots)\sigma_x\sigma_y \\ &= \ldots\ldots\ldots\sigma + \ldots\ldots\ldots\sigma_y^2 - \ldots\ldots\ldots\sigma_y^2 + \ldots\ldots\ldots\sigma_y^2 \\ &\quad + \ldots\ldots\ldots r_{x,y}x_x\sigma_y - \ldots\ldots\ldots r_{x,y}x_x\sigma_y \end{aligned}$$
- at $a*$ minimum variance portfolio ; $\dfrac{dVAR_p}{da} = 0$

The Minimum Variance Portfolio (11)

- $$\begin{aligned}\frac{dVAR_p}{da} &= \ldots\ldots\sigma_x^2 - \ldots\ldots\ldots\sigma_y^2 + \ldots\ldots\ldots\ldots\sigma_y^2 + \ldots\ldots\ldots\ldots\ldots r_{x.y}\sigma_x\sigma_y - \ldots\\ &= \quad a(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$
- $a^* = \dfrac{\sigma_y^2 - r_{x,y}\sigma_x\sigma_y}{\sigma_x^2 + \sigma_y^2 - 2r_{x,y}\sigma_x\sigma_y}$
- from the example;

X	$E(R_x) = 10$	$\sigma_x^2 = 0.0076$	$\sigma_x = 0.0872$
Y	$E(R_y) = 8$	$\sigma_y^2 = 0.00708$	$\sigma_y = 0.0841$
X,Y	$r_{x,y} = -0.33$		

- $a^* = \dfrac{0.0078 - (-0.33)(0.0872)(0.0841)}{0.0076 + 0.00708 - 2(-0.33)(0.0872)(0.0841)} =$ 0.487
- (to be continuted..)