Quiz 2: Date: May 5, 2022 from 11.00-12.30

Question 1 (40 marks) Score....................

Consider the Muliperiod model of consumption and portfolio choice. Let an individual in this economy has the utility function as follow:

$U(C_s, s)$ B

$$\max_{C_s,\omega_s,\forall_t} E_t\left[\sum_{s=t}^{T-1}\delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)+\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

Assume that there is no wage income $(y_t = 0\,\forall t)$ and a constant risk-free rate return asset , $R_{ft} = R_f$. Also asssume that n=1 and the return of a single risky asset, R_{rt},is independently and identically distributed over time. Denote the proportion of wealth invested in the risky asset at date t as ω_t.

Please read and answer the following questions carefully and completely.

Score.....................

Question 1.1 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-1, C^*_{T-1} and ω^*_{T-1}., and give an explicit expression for C^*_{T-1}

$$\max_{C^*_{T-1}} E_t\left[\sum_{s=t}^{T-1} \delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)+\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

$$U_C(C_{s-1}, s-1) - E_{T-1}\left[B_W(W_T, T)R_{T-1}\right] = 0$$

$$U_C(C_{s-1}, s-1) = E_{T-1}\left[B_W(W_T, T)R_{T-1}\right.$$

$$\frac{\partial C_s^{(1-\gamma)}}{(1-\gamma)} \to \frac{C_s^{(1-\gamma)-1}}{\cancel{1-\gamma}}\cancel{(1-\gamma)}$$

$$= C_s^{-\gamma}$$

$$\delta^{s-1} C_{s-1}^{-\gamma} = E_{T-1}\left(\delta^T W_T^{-\gamma} R_{T-1}\right) \quad ; W_T = S_{T-1} R_{T-1}$$

$$\delta^{s-1} C_{s-1}^{-\gamma} = E_{T-1}\left(\frac{\delta^T R_{T-1}}{W_T^{\gamma}}\right)$$

$$\delta^{s-1}\frac{1}{C_{s-1}^{\gamma}} = \frac{\delta^T R_{T-1}}{W_T^{\gamma}} \quad ; S = T$$

$$\delta^{T-1}\frac{1}{C_{T-1}^{*\gamma}} = \frac{\delta^T R_{T-1}}{W^{\gamma}_T} \quad — \quad 1$$

$$C_{T-1}^{*\gamma} = \frac{W^{\gamma}_T}{\delta R_{T-1}}$$

$$\max_{w_{i,T-1}} E_t\left[\sum_{s=t}^{T-1} \delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)+\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

$$E_{T-1}\left[B_W(W_T, T)\left(R_{i,T-1} - R_{f,T-1}\right)\right. = 0 \quad i = 1, \ldots, n$$

$$E_{T-1}\left[\delta^T W_T^{-\gamma}\left(R_{i,T-1} - R_{f,T-1}\right)\right. = 0 \quad , n = 1, \; i = 1 \quad — \quad 2$$

$$2 = 1$$

$$\delta^{T-1} C_{T-1}^{*-\gamma} = E_{T-1}\left[\delta^T W_T^{-\gamma}\left(R_{f,T-1} + \sum_{i=1}^{n} w_{i,T-1}\left(R_{i,T-1} - R_{f,T-1}\right)\right)\right]$$

$$\delta^{T-1} C_{T-1}^{*-\gamma} = R_{f,T-1} E_{T-1}\left(\delta^T W_T^{*\gamma}\right]$$

$$C_{T-1}^{*-\gamma} = \frac{R_{f,T-1}\,\delta^T W_T^{*\gamma}}{\delta^{T-1}}$$

$$C_{T-1}^{*\gamma} = \frac{\delta^{T-1}}{R_{f,T-1}\,\delta^T W_T^{*\gamma}}$$

$$C^*_{T-1} = \sqrt[\gamma]{\frac{\delta^{T-1}}{R_{f,T-1}\,\delta^T W_T^{*\gamma}}}$$

$$= \sqrt[\gamma]{\frac{1}{R_{f,T-1}\,\delta W_T^{*\gamma}}}$$

Score.....................

Question 1.2 (10 marks) Solve for the form of J(W_{T-1},T-1).

$$J(W_{T-1}, T-1) = U_C \frac{\partial C^*_{T-1}}{\partial W_{T-1}} - E_{T-1}[B_{W_T} R_{T-1}] \frac{\partial C^*_{T-1}}{\partial W_{T-1}} + E_{T-1}(B_{W_T} R_{T-1}]$$

$$E_{T-1} = B_{W_T}(R_{i,T-1} - R_{f,T-1}] = 0 \qquad U_C = E_{T-1}[R_{W_T} R_{T-1}]$$

$$J(W_{T-1}, T-1) = U_C(C^*_{T-1}, T-1)$$

$$= \frac{W^{\gamma}_T}{\delta R_{T-1}}$$

//

$$U_c(C^*_{T-1}, T-1) = E_{T-1}(B_W(W_T, T)(R_{T-1})$$

$$\delta^{T-1} \frac{1}{C^{*\gamma}_{T-1}} = \frac{\delta^T R_{T-1}}{W^{\gamma}_T}$$

$$C^{*\gamma}_{T-1} = \frac{W^{\gamma}_T}{\delta R_{T-1}}$$

Score.....................

Question 1.3 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-2, C^*_{T-2} and ω^*_{T-2}., and give an explicit expression for C^*_{T-2}

$J(W_{T-2}, T-2) = \max U(C_{T-2}, T-2) + E_{T-2}[J(W_{T-1}, T-1)]$

$= U(C_{T-2}, T-2)$

$\delta^{T-2} \frac{1}{C^{*\gamma}_{T-2}} = \frac{\delta^T R_{T-2}}{W^\gamma_T}$

$C^{*\gamma}_{T-2} = \frac{W^\gamma_T}{\delta R_{T-2}}$ //

$J(W_{T-2}, T-2 = \frac{W^\gamma_T}{\delta R_{T-2}}$ //

Score......................

Question 1.4 (10 marks) Solve for the form of J(W_{T-2},T-2).Based on the pattern for T-1 and T-2, provide expressions for the optimal consumption and portfolio weight at any date T-t, t=1,2,3,....