Tutorial 3
Matrix Algebra

1. Find all solution to the following system of equations

   (a)

$$\begin{aligned}3x+4y+z&=1\\2x+3y&=0\\4x+3y-z&=-2\end{aligned}$$

(Ans: x=-3/7, y=2/7,z=8/7)

   (b)

$$\begin{aligned}x+2y-z&=2\\2x+5y+2z&=-1\\7x+17y+5z&=-1\end{aligned}$$

Ans:

$$\begin{aligned}x&=12+9z\\y&=-5-4z\end{aligned}$$ (parametric form)

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}12\\-5\\0\end{bmatrix}+C\begin{bmatrix}9\\-4\\1\end{bmatrix}$$; C∈R (vectors form)

   c)

$$\begin{aligned}x+10z&=5\\3x+y-4z&=-1\\4x+y+6z&=1\end{aligned}$$

(ans: No solution)

2. Compute the rank of $A=\begin{bmatrix}1&1&-1&4\\2&1&3&0\\0&1&-5&8\end{bmatrix}$

(ans: 2)

3. Find a condition on the number *a,b,c* such that the following system of equations is consistent. When that condition is satisfied, find all solutions (in term of *a,b, c*)

$$\begin{aligned}x+3y+z&=a\\-x-2y+z&=b\\3x+7y\text{-}z&=c\end{aligned}$$

Ans: Consistent if c=a-2b

$$x = 5z - (2a + 3b)$$
$$y = (a + b) - 2z \quad \text{(Pararametric form)}$$
$$\text{Z} = \text{free variables}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -(2a+3b) \\ a+b \\ 0 \end{bmatrix} + C \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \quad C \in R \text{ (vectors form)}$$

4. Suppose a square matrix A satisfies $A = 2A^T$. Show that necessarily *A=0*.

Ans:

$$A = 2(2A^T)^T = 2(2(A^T)^T = 4A$$
$$3A = 0$$
$$A = 0$$

5. In each case find the matrix A

(a) $\left(3A^T + 2\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right)^T = \begin{bmatrix} 8 & 0 \\ 3 & 1 \end{bmatrix}$ Ans: $A = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$

(b) $\left(2A^T - 5\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}\right)^T = 4A - 9\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ Ans: $A = \begin{bmatrix} 2 & 7 \\ \frac{-9}{2} & -5 \end{bmatrix}$

6. Find the inverse of the matrix

$A = \begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{bmatrix}$ Ans: $A^{-1} = \frac{1}{2}\begin{bmatrix} -3 & -3 & 11 \\ 1 & 1 & -3 \\ 1 & -1 & -1 \end{bmatrix}$

7. Find A if $\left(A^T - 2I\right)^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ Ans: $A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$

8. Suppose A is 3 by 5, B is 5 by 3, C is 5 by 1, and D is 3 by 1. Which of these matrix operations are allowed, and what are the shape of the results?

BA, A(B+C), ABD, AC+BD, ABABD

(Ans: **BA** (a 5 x5 matrix) **ABD** (a 3x1 matrix) and **ABABD** (a 3x1 matrix))

9. Compute AB where $A=\begin{bmatrix} 2 & 0 & -3 \\ -1 & 6 & 5 \end{bmatrix}$ and $B=\begin{bmatrix} 0 & 3 \\ -4 & 7 \\ 2 & -5 \end{bmatrix}$. Show how you obtain the result using all three methods you study in the class.

Ans: $\begin{bmatrix} -6 & 21 \\ -14 & 14 \end{bmatrix}$

10. Find all solutions to the following system of linear equations:

$$x+y-3z=3$$
$$-2x-y=-4$$
$$4x+2y+3z=7$$

Ans: x=2, y=0,z = -1/3

11. Determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system

a) $\left[\begin{array}{cc|c} 2 & 3 & h \\ 4 & 6 & 7 \end{array}\right]$ (Ans: *h*=7/2)   b) $\left[\begin{array}{cc|c} 1 & -3 & 2 \\ 5 & h & -7 \end{array}\right]$ (Ans: $h \neq -15$)

12. For matrix

$$\mathbf{A}=\begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix}$$

determine the echelon form U, the basic variables, the free variables and the general solution to $\mathbf{A}x=0$. Then apply, elimination to $\mathrm{A}x=\mathbf{b}$, with b_1 and b_2 on the right side; find the conditions for $\mathrm{A}x=\mathbf{b}$ to be consistent and find the general solution. What is the rank of $\mathbf{A}$?

Ans: For $\mathbf{A}x=0$ ➔ $\mathbf{x}=C_1\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}+C_3\begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}+C_4\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ where $C_1, C_3, C_4 \in \mathrm{R}$

For $\mathbf{A}x=\mathbf{b}$ ➔

$$\mathbf{x}=\begin{bmatrix} 0 \\ b_1 \\ 0 \\ 0 \end{bmatrix}+C_1\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}+C_3\begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}+C_4\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ where } C_1, C_3, C_4 \in \mathrm{R}$$

13. a) Find all solutions to

$$\mathbf{Ux} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Ans: $\mathbf{x} = C_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \text{where } C_1, C_2 \in \mathrm{R}$

b) If the right side is changed from $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$, what are the solutions?

Ans: $\mathbf{x} = \begin{bmatrix} a-3b \\ 1 \\ b \\ 0 \end{bmatrix} + C_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \text{where } C_1, C_2 \in \mathrm{R}$

14. Use row operation to find

$$\begin{aligned} 6x_1 + x_2 + x_3 &= 6 \\ 5x_1 + x_2 + 2x_3 &= 4 \\ 4x_1 + x_2 - x_3 &= -2 \end{aligned}$$

Ans: $\begin{bmatrix} 3 \\ -13 \\ 1 \end{bmatrix}$

15. Suppose

$Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ has no solution

but

$Ax = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ has infinitely many solutions.

a) Find all possible information about r, m and n. (r is the rank and A is of the size m×n)

b) Find an example of such a matrix A with r, m and n all as small as possible.

> Ans:
>
> a) m=3, 0<r<m and r<n
>
> b) r=1, m=3, n=2; $A = \begin{bmatrix} 3 & 0 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}$

16. a) By elimination put A into its upper triangular form U. Which are the pivot columns and free columns?

$$A = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 2 & 8 & 5 & 2 \\ 1 & 5 & 3 & 1 \end{bmatrix}$$

(Ans: pivot columns 1,2 and free columns 3,4)

b) Describe specifically the vectors which are solutions to Ax=0

c) Does Ax=b have a solution for the right side b $= \begin{bmatrix} 3 \\ 8 \\ 5 \end{bmatrix}$? If it does, find one particular solution and then complete solution to this system Ax=b.

> Ans:
>
> $$x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

17. a) Show that $A = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ has inverse $C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

Show that $A = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ has inverse $C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

We verify that $AC = I$ and $CA = I$:

$$AC = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and

$$CA = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

b) Show that $A = \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix}$ has no inverse.

If $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is *any* matrix, then $AC = \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a-3c & 2b-3d \\ 0 & 0 \end{bmatrix}$. Since the (2, 2)-entry of AC is not 1, AC can never equal I for any choice of the matrix C.

18. a) Find the inverse of the matrix $A = \begin{bmatrix} 1 & 4 & -1 \\ 2 & 7 & 1 \\ 1 & 3 & 0 \end{bmatrix}$ using Gauss-Jordan method.

$$A^{-1} = \begin{bmatrix} -\frac{3}{2} & -\frac{3}{2} & \frac{11}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -3 & -3 & 11 \\ 1 & 1 & -3 \\ -1 & 1 & -1 \end{bmatrix}$$

b) Show that the matrix $A = \begin{bmatrix} 1 & 3 & -2 \\ 1 & 2 & 0 \\ 2 & 8 & -8 \end{bmatrix}$ is not invertible.

19. Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

(a) Use Gauss-Jordan elimination to compute the inverse of the matrix A.

Ans:

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & -2 & -1 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

(b) Use the answer in part (a) to solve the system $Ax = b$.

Ans:

$$\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

20. Compute the determinant of

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 0 & 0 \\ 0 & 7 & 8 & 0 \\ 0 & 0 & 9 & 10 \end{bmatrix}.$$

(Ans: -530)