# Macroeconomics

**Lecture 4**

# Example

$$\max \quad \sum_{t=0}^{\infty} \beta^t U(c_t), \qquad 0<\beta<1$$

*Subject to*

$$A_{t+1} = R_t (A_t + y_t - c_t),$$

$$or, \quad c_t + \sum_{j=1}^{\infty} \left( \prod_{k=0}^{j-1} R_{t+k}^{-1} \right) c_{t+j} = y_t + \sum_{j=1}^{\infty} \left( \prod_{k=0}^{j-1} R_{t+k}^{-1} \right) y_{t+j} + A_t . \qquad (10)$$

$$A_0 \ \ given,$$

- State variables are $A_t$ , $y_t$, $R_{t-1}$.
- Control variables is savings, $s_t = A_t + y_t - c_t$
- Transition function $A_{t+1} = R_t.(A_t+y_t-c_t)$ does not involve state variable. (in this case, we have $x_{t+1}=g(c_t)$, instead of $x_{t+1}=g(x_t, c_t)$)

# Saving under certainty

- Bellman's equation

$$V\left(A_t, y_t, R_{t-1}\right) = \max_{s_t}\left[U\left(A_t + y_t - s_t\right) + \beta V\left(R_t s_t, y_{t+1}, R_t\right)\right]$$

$$where \qquad s_t = R_t^{-1} A_{t+1},$$

- Note that $c_t = A_t + y_t - s_t$ , or $s_t = A_t + y_t - c_t$. Hence by recalling (9), which is the same marginal condition as (1.4b), and suppose that the optimal choice $s_t = h(A_t, y_t, R_{t-1})$ then we have

$$\frac{\partial V\left(A_t, y_t, R_{t-1}\right)}{\partial A_t} = \frac{\partial U\left(c_t\right)}{\partial c_t}\frac{\partial\left(A_t + y_t - h\left(A_t, y_t, R_{t-1}\right)\right)}{\partial A_t}$$

$$+\beta\frac{\partial V\left(R_t h\left(A_t, y_t, R_{t-1}\right), y_{t+1}, R_t\right)}{\partial R_t h\left(A_t, y_t, R_{t-1}\right)}.R_t.\frac{\partial h\left(A_t, y_t, R_{t-1}\right)}{\partial A_t},$$

# Saving under certainty

$$\frac{\partial V\left(A_t, y_t, R_{t-1}\right)}{\partial A_t} = \frac{\partial U\left(c_t\right)}{\partial c_t}\left[1 - \frac{\partial h\left(A_t, y_t, R_{t-1}\right)}{\partial A_t}\right]$$

$$+\beta \frac{\partial V\left(R_t h\left(A_t, y_t, R_{t-1}\right), y_{t+1}, R_t\right)}{\partial R_t h\left(A_t, y_t, R_{t-1}\right)} R_t \frac{\partial h\left(A_t, y_t, R_{t-1}\right)}{\partial A_t},$$

$$= U'\left(c_t\right) - \frac{\partial h\left(A_t, y_t, R_{t-1}\right)}{\partial A_t}\left[U'\left(c_t\right) - \beta V'\left(A_{t+1}, y_{t+1}, R_t\right) R_t\right],$$

$$= U'\left(c_t\right), \qquad \left(\because First - order\ condition\ w.r.t.\ u_t\right)$$

*The above result also valid for* $t+1$, *hence,*

$$\frac{\partial V\left(A_{t+1}, y_{t+1}, R_t\right)}{\partial A_{t+1}} = \frac{\partial U\left(c_{t+1}\right)}{\partial c_{t+1}} = U'\left(c_{t+1}\right), \qquad (11)$$

$Next,\ by\ differentiating\ Bellman\ equation\ w.r.t.\ s_t,\ one\ has$

$$\frac{\partial U\left(A_t + y_t - s_t\right)}{\partial\left(A_t + y_t - s_t\right)} \frac{\partial\left(A_t + y_t - s_t\right)}{\partial s_t} + \beta \frac{\partial V\left(s_t R_t, y_{t+1}, R_t\right)}{\partial s_t R_t} \frac{\partial s_t R_t}{\partial s_t} = 0,$$

$$-U'\left(c_t\right) + \beta V'\left(A_{t+1}, y_{t+1}, R_t\right) R_t = 0,$$

$$-U'\left(c_t\right) + \beta R_t U'\left(c_{t+1}\right) = 0, \qquad \left[\because Eq.11\right] \qquad (12)$$

$Eq.12\ is\ the\ Euler\ equation\ for\ s_t,$

$If \;\; U\left(c_t\right) = \ln c_t, \;\; then \;\; from \;\; (12)$

$c_{t+j} = \beta^j \left[ R_t . R_{t+1} .... R_{t+j-1} \right] c_t$

$Substituing \;\; this \; \text{int} o \; the \; left \; hand \;\; side \; of \;\; (10), \; which \;\; is$

$c_t + \sum_{j=1}^{\infty} \left( \prod_{k=0}^{j-1} R_{t+k}^{-1} \right) c_{t+j}, \;\; then$

$$
\begin{aligned}
c_t + \sum_{j=1}^{\infty} \left( \prod_{k=0}^{j-1} R_{t+k}^{-1} \right) c_{t+j} &= c_t + \sum_{j=1}^{\infty} \left[ \left( \prod_{k=0}^{j-1} R_{t+k}^{-1} \right) \left( \beta^j \left[ R_t . R_{t+1} .... R_{t+j-1} \right] c_t \right) \right] \\
&= c_t \left[ 1 + \sum_{j=1}^{\infty} \left[ \left( \prod_{k=0}^{j-1} R_{t+k}^{-1} \right) \left( \beta^j \left[ R_t . R_{t+1} .... R_{t+j-1} \right] \right) \right] \right] \\
&= c_t \left[ 1 + \left\{ \beta + \beta^2 + .......... \right\} \right] \\
&= c_t \left(1 - \beta\right)^{-1}
\end{aligned}
$$

$Hence, \; (10) \; and \; (12) \; imply \; that$

$$
c_t = \left(1 - \beta\right) \left[ y_t + \sum_{j=1}^{\infty} \left( \prod_{k=0}^{j-1} R_{t+k}^{-1} \right) y_{t+j} + A_t \right] \qquad (13)
$$

# Optimal Growth under certainty

$$\max \quad \sum_{t=0}^{\infty} \beta^t U(c_t), \qquad 0 < \beta < 1$$

$$s.t. \qquad c_t + k_{t+1} = f(k_t), \qquad k_0 > 0 \;\; given, \quad c_t \geq 0$$

$$U'(0) = +\infty, \; U' > 0, \; U'' < 0,$$

$$f'(0) = \infty, \; f'(\infty) = 0, \; f' > 0, f'' < 0.$$

$$(\therefore \; one \; may \; write \;\; k_{t+1} = f(k_t) - c_t = g(c_t, k_t))$$

# Optimal Growth under certainty

$$\textit{Bellman's equation},$$

$$V(k_t) = \max_{k_{t+1}} \left\{ U\left[ f(k_t) - k_{t+1} \right] + \beta V(k_{t+1}) \right\}$$

$$\textit{Euler equation for } k_{t+1},$$

$$\frac{\partial U\left[ f(k_t) - k_{t+1} \right]}{\partial \left[ f(k_t) - k_{t+1} \right]} \frac{d\left[ f(k_t) - k_{t+1} \right]}{d\left[ k_{t+1} \right]} + \beta \frac{\partial V(k_{t+1})}{\partial k_{t+1}} = 0,$$

$$\frac{\partial U(c_t)}{\partial c_t}(-1) + \beta \frac{\partial V(k_{t+1})}{\partial k_{t+1}} = 0,$$

$$\frac{\partial U(c_t)}{\partial c_t} = \beta \frac{\partial V(k_{t+1})}{\partial k_{t+1}}, \qquad (14)$$

# Optimal Growth

*From* (9), (*The* marginal *condition*), *and the optimal choice* $k_{t+1} = h(k_t)$

$$V'(k_t) = \frac{\partial U[f(k_t) - h(k_t)]}{\partial[f(k_t) - h(k_t)]}\left[\frac{df(k_t)}{dk_t} - \frac{dh(k_t)}{dk_t}\right] + \beta \frac{\partial V(h(k_t))}{\partial h(k_t)} \frac{dh(k_t)}{dk_t}$$

$$= \frac{\partial U(c_t)}{\partial c_t}[f'(k_t)] - \frac{\partial h(k_t)}{k_t}\left[\frac{\partial U(c_t)}{\partial c_t} - \beta V'(k_{t+1})\right]$$

$$= \frac{\partial U(c_t)}{\partial c_t}[f'(k_t)], \qquad (\because eq.(14))$$

*The above result is also valid at* $t+1$, *hence*

$$V'(k_{t+1}) = \frac{\partial U(c_{t+1})}{\partial c_{t+1}}[f'(k_{t+1})], \qquad (15)$$

*Hence, substitute* (15) into (14), *one has*

$$-U'(c_t) + \beta U'(c_{t+1}) f'(k_{t+1}) = 0 \qquad (16)$$

# Optimal Growth

$$\textit{From } (16), \qquad f'(k_{t+1}) = \frac{U'(c_t)}{\beta U'(c_{t+1})},$$

$$= \frac{U'(f(k_t) - k_{t+1})}{\beta U'(f(k_{t+1}) - k_{t+2})}.$$

$\therefore$ *Optimal policy function* $k_{t+2} = h(k_{t+1})$ *must be a non decreasing function of* $k_{t+1}$. *This is becasue*

$$f''(k_{t+1}) = \frac{[\beta U'(c_{t+1})][U''(c_t).(-1)] - [U'(c_t)]\left[\beta U''(c_{t+1}).\left(f'(k_{t+1}) - \frac{\partial h(k_{t+1})}{\partial k_{t+1}}\right)\right]}{[\beta U'(c_{t+1})]^2} \leq 0$$

$\Rightarrow$ *Suppose that* $\frac{\partial h(k_{t+1})}{\partial k_{t+1}} < 0$, *then the RHS of the above equation will be greater than zero, which is contradicted to* $f''(k_{t+1}) \leq 0$. *Hence, it must be that*

$$\frac{\partial h(k_{t+1})}{\partial k_{t+1}} \geq 0.$$

# Appendix

$$A_{t+1} = R_t\left(A_t + y_t - c_t\right) \qquad (a)$$

$$A_{t+2} = R_{t+1}\left(A_{t+1} + y_{t+1} - c_{t+1}\right) \qquad (b)$$

$$A_{t+3} = R_{t+2}\left(A_{t+2} + y_{t+2} - c_{t+2}\right) \qquad (c)$$

*Put* $(a)$ into $(b)$

$$A_{t+2} = R_{t+1}\left(R_t\left[A_t + y_t - c_t\right] + y_{t+1} - c_{t+1}\right) \qquad (d)$$

*Put* $(d)$ into $(c)$

$$A_{t+3} = R_{t+2}\left(R_{t+1}\left(R_t\left[A_t + y_t - c_t\right] + y_{t+1} - c_{t+1}\right) + y_{t+2} - c_{t+2}\right)$$

$$A_{t+3} = R_{t+2}R_{t+1}R_t\left[A_t + y_t - c_t\right] + R_{t+2}R_{t+1}\left[y_{t+1} - c_{t+1}\right] + R_{t+2}\left[y_{t+2} - c_{t+2}\right]$$

$$A_{t+3} = \left[\prod_{k=0}^{2} R_{t+k}\right]\left[A_t + y_t - c_t\right] + \left[\prod_{k=1}^{2} R_{t+k}\right]\left[y_{t+1} - c_{t+1}\right] + R_{t+2}\left[y_{t+2} - c_{t+2}\right]$$

$$A_{t+j} = \left[\prod_{k=0}^{j-1} R_{t+k}\right]\left[A_t + y_t - c_t\right] + \left[\prod_{k=1}^{j-1} R_{t+k}\right]\left[y_{t+1} - c_{t+1}\right] + \left[\prod_{k=2}^{j-1} R_{t+k}\right]\left[y_{t+2} - c_{t+2}\right]$$

$$+ \dots + \left[\prod_{k=j-2}^{j-1} R_{t+k}\right]\left[y_{t+j-2} - c_{t+j-2}\right] + R_{t+j-1}\left[y_{t+j-1} - c_{t+j-1}\right] \qquad (e)$$

$$Let \quad j \to \infty, \qquad \lim_{j\to\infty} A_{t+j} = 0, \;\; and \quad eq.(e) \;\; becomes$$

$$0 = \left[\prod_{k=0}^{\infty} R_{t+k}\right]\left[A_t + y_t - c_t\right] + \left[\prod_{k=1}^{\infty} R_{t+k}\right]\left[y_{t+1} - c_{t+1}\right]$$

$$+ .... + \left[\prod_{k=2}^{\infty} R_{t+k}\right]\left[y_{t+j-2} - c_{t+j-2}\right] + R_{t+j-1}\left[y_{t+j-1} - c_{t+j-1}\right], \quad .......(f)$$

*Or*,

$$\left[\prod_{k=0}^{\infty} R_{t+k}\right]c_t + \left[\prod_{k=1}^{\infty} R_{t+k}\right]c_{t+1} + \left[\prod_{k=2}^{\infty} R_{t+k}\right]c_{t+2} + ..... + \left[\prod_{k=j-1}^{\infty} R_{t+k}\right]c_{t+j-1} + ....$$

$$= \left[\prod_{k=0}^{\infty} R_{t+k}\right]y_t + \left[\prod_{k=1}^{\infty} R_{t+k}\right]y_{t+1} + ..... + \left[\prod_{k=j-1}^{\infty} R_{t+k}\right]y_{t+j-1} + ... + \left[\prod_{k=0}^{\infty} R_{t+k}\right]A_t, \quad .......(g)$$

$$Then, multiplying \; both \; sides \; of \;\; eq.(g) \; by \left[\prod_{k=0}^{\infty} R_{t+k}\right]^{-1},$$

$$c_t + (R_t)^{-1} c_{t+1} + (R_t.R_{t+1})^{-1} c_{t+2} + ... + (R_t.R_{t+1}...R_{t+j-1})^{-1} c_{t+j} + ...$$

$$= y_t + (R_t)^{-1} y_{t+1} + (R_t.R_{t+1})^{-1} y_{t+2} + ....... + (R_t.R_{t+1}...R_{t+j-1})^{-1} y_{t+j} + ..... + A_t,$$

*Hence*,

$$c_t + \sum_{j=1}^{\infty}\left(\prod_{k=0}^{j-1} R_{t+k}^{-1}\right)c_{t+j} = y_t + \sum_{j=1}^{\infty}\left(\prod_{k=0}^{j-1} R_{t+k}^{-1}\right)y_{t+j} + A_t, \qquad (1.10) \qquad Q.E.D.$$