Chapter 2
Calculus of Single Variable

Here we will discuss only the calculus of optimization of a twice-differentiable single-variable function $y = f(x)$. See also Appendix to Chapter 1 of Baldani, et. al. [2005], p. 22-36.

2.1 Derivative

Definition Let $f : \boldsymbol{S} \to \boldsymbol{R}$, $\boldsymbol{S} \subseteq \boldsymbol{R}$, be a function. The ***derivative*** of the function *f* at *x*, $x \in \boldsymbol{S}$, is defined as ,

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

• The derivative exists if, and only if, the limit does.

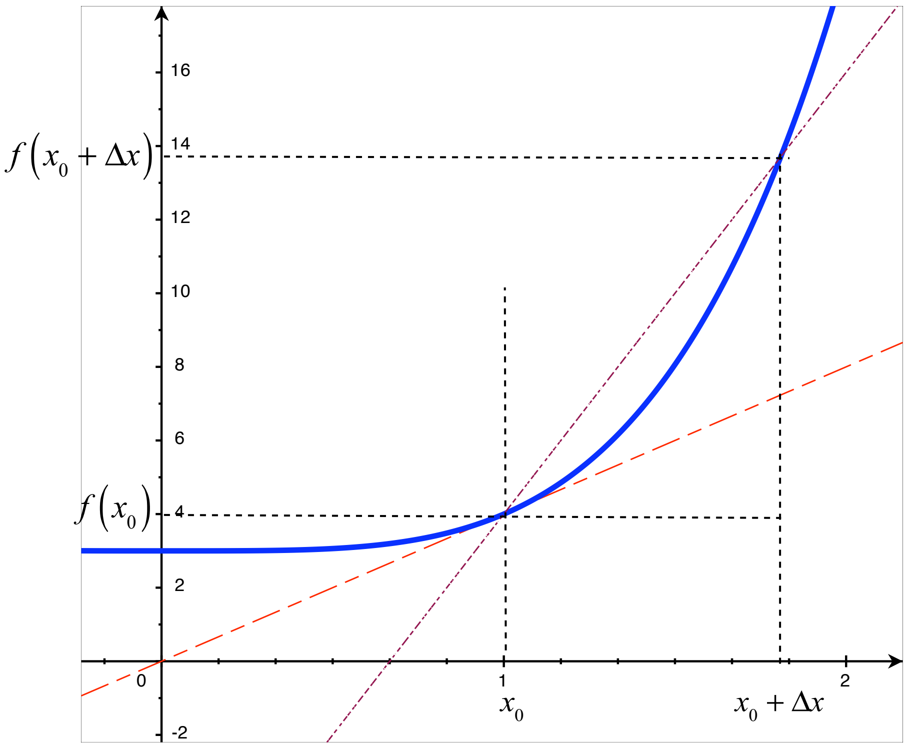

Figure 2.1 Derivative of *f* at x_0

• *f* is said to be differentiable at x_0 if $f'(x_0)$ exists.

• *f* is said to be differentiable if it is differentiable at each $x \in \boldsymbol{S}$, and we can write $f \in \boldsymbol{C}^1$, where $\boldsymbol{C}^1$ is the set of all differentiable functions.

- $f'(x)$ is interpreted as the ***slope*** or ***instantaneous rate of change*** of the function *f* at $x \in \boldsymbol{S}$.
- $f'(x)$ is also a function from the same domain $\boldsymbol{S} \subseteq \boldsymbol{R}$ to $\boldsymbol{R}$.
- If $y = f(x)$, we can write $f'(x) = \frac{dy}{dx} = \frac{df(x)}{dx}$.

The followings are formulae of derivative frequently used.

a) If $f(x) = ax^n$, then $f'(x) = anx^{n-1}$.

b) If $f(x) = \sum_{i=1}^{n} g_i(x)$, then $f'(x) = \sum_{i=1}^{n} g_i'(x)$.

c) If $f(x) = g(x)h(x)$, then

$$f'(x) = g(x)h'(x) + h(x)g'(x).$$

d) If $f(x) = \frac{g(x)}{h(x)}$, $h(x) \neq 0$ then

$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{\left[h(x)\right]^2}.$$

e) If $f(x) = \boldsymbol{ln}(x)$, then $f'(x) = \frac{1}{x}$.

f) If $f(x) = e^x$, then $f'(x) = e^x$.

g) If $f(x) = \boldsymbol{log}_a(x)$, then

$$f'(x) = \frac{1}{x}\boldsymbol{log}_a e, a > 0, a \neq 1.$$

h) If $f(x) = a^x$, then $f'(x) = a^x \boldsymbol{ln}\, a$, $a > 0$.

i) Chain Rule: If $z = h(x) = g(f(x))$, $y = f(x)$ then

$$h'(x) = g'(f(x))f'(x)$$

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}.$$

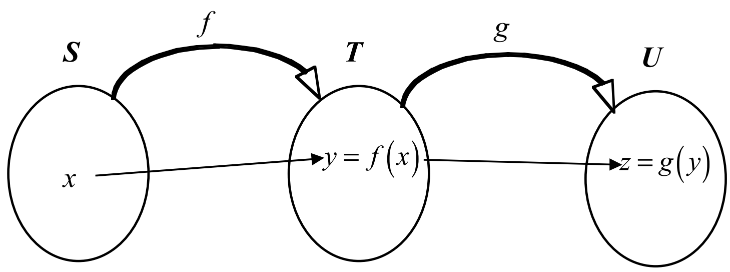

Figure 2.2 Diagram for a composite function

Example If $h(x) = \boldsymbol{ln}(f(x))$, then

$$h'(x) = \frac{1}{f(x)} f'(x).$$

Example If $h(x) = e^{f(x)}$, then $h'(x) = e^{f(x)} f'(x)$.

Example Let $z = y^5$, and $y = (1 - x^3)$. The derivative of *y* with respect to *x* is given by,

$$\begin{aligned}\frac{dz}{dx} &= \frac{dz}{dy}\frac{dy}{dx} \\ &= \frac{d}{dy}(y^5)\frac{d}{dx}(1-x^3) \\ &= 5y^4(-3x^2) \\ &= 5(1-x^3)^4(-3x^2).\end{aligned}$$

HW Find the derivative of the following functions.

a) $s = (t^2 - 3)^4$

b) $s = \boldsymbol{ln}(t^2 - 3)$

c) $z = \dfrac{3}{(1 - y^2)^{0.4}}$

d) $y = (x^2 + 4)^2 \boldsymbol{log}(2x^3 - 1)$

e) $y = 1 + x + x^2 + \cdots + x^n$

f) $PV = A\left[1 + \dfrac{1}{1+r} + \dfrac{1}{(1+r)^2} + \cdots + \dfrac{1}{(1+r)^n}\right]$

g) $F = Ae^{rt}$

2.2 Examples of Derivatives in Economics

Example: The derivative of the consumption function $C = a + bY$ is just $\frac{dC}{dY} = b$.

Example: From the budget line function, $p_x x + p_y y = B$, we can write $y = \frac{B}{p_y} - \frac{p_x}{p_y}x$, and thus $\frac{dy}{dx} = -\frac{p_x}{p_y}$.

Example: Suppose the total product as a function of L is given by, $TP(L) = 2L^{\frac{2}{3}}$, the marginal product which is the slope of total product will be just

$$MP(L) = \frac{dTP(L)}{dL} = \frac{4}{3}L^{-\frac{1}{3}}.$$

Example: Taking the derivative on both sides of the equation $TP(L) = AP(L) \times L$, we have

$$\begin{aligned} MP(L) &= AP(L)\frac{dL}{dL} + L\frac{dAP(L)}{dL} \\ &= AP(L) + L\frac{dAP(L)}{dL}. \end{aligned}$$

HW Can you get the same relation between $MP(L)$ and $AP(L)$ by taking the derivative on both sides of the equation $AP(L) = \frac{TP(L)}{L}$?

Example: If the demand $p = f(q)$ is a linear funcion, the marginal revenue curve is also linear with slope twice that of the demand function, in absolute value. Since price is by definition the average revenue, we can write

$$p = AR(q) = a - bq$$
$$TR(q) = AR(q) \times q = aq - bq^2$$
$$MR(q) = \frac{dTR(q)}{dq} = a - 2bq$$

Example: Elasticity of demand

$$\frac{\%\Delta Q}{\%\Delta P} = \frac{\Delta Q}{\Delta P}\frac{P}{Q}$$
$$= \frac{dQ}{dP}\frac{P}{Q}.$$

Example: The elasticity of demand can be shown to be $\frac{d\,\ln Q}{d\,\ln P} = \frac{dQ}{dP}\frac{P}{Q}$.

Proof Let $r = \ln P$. So $P = e^r$. By chain rule,

$$\frac{d\,\ln Q}{d\,\ln P} = \frac{d\,\ln Q}{dP}\frac{dP}{d\,\ln P} = \frac{d\,\ln Q}{dP}\frac{dP}{dr}$$
$$= \frac{1}{Q}\frac{dQ}{dP}e^r$$
$$= \frac{dQ}{dP}\frac{P}{Q}.$$

❑

HW Show that the elasticity can also be given as $\frac{d\,\log_a Q}{d\,\log_a P}$ for any base *a*.

Example: If a demand function is given by $Q = P^{-2}$, we have

$$\ln Q = -2\ln P$$
$$\frac{d\,\ln Q}{d\,\ln P} = -2.$$

That is, this demand function has a constant elasticity being -2 at any price.

Example Continuous time interest rate as an extension of discrete time case.

In discrete time case,

$$Y(t+1)=(1+r)Y(t)\Rightarrow Y(t)=Y(0)(1+r)^{t}.$$

Analogously, in continuous time

$$\begin{aligned}&Y(t+1)=(1+r)Y(t)\\ &\qquad\Rightarrow Y(t+1)-Y(t)=rY(t)\\ &\qquad\Rightarrow \frac{Y(t+\Delta t)-Y(t)}{\Delta t}=iY(t).\end{aligned}$$

We have $\lim\limits_{\Delta t\to 0}\dfrac{Y(t+\Delta t)-Y(t)}{\Delta t}=Y'(t)=iY(t)$.

Theorem: From $Y'(t)=iY(t)$, we have by solving a simple differential equation

a) $Y(t)=Y(0)e^{it}$.

b) $i=\ln(1+r)$

2.3 Derivatives and Increasing Functions

Definition A function $f:\boldsymbol{S}\to\boldsymbol{R}$, $\boldsymbol{S}\subseteq\boldsymbol{R}$ is ***increasing*** in $\boldsymbol{A}$, $\boldsymbol{A}\subseteq\boldsymbol{S}$, if for any $x_1,x_2\in\boldsymbol{A}$,

$$x_1\le x_2\Leftrightarrow f(x_1)\le f(x_2).$$

• *f* is increasing whereever $f'(x)\ge 0$.

Problem Can you define ***strictly increasing*** function and give the relation to its derivative? What about ***decreasing*** and ***strictly decreasing*** functions?

Example The function $f(x)=2x^3+x-2$ is always increasing.

Example The function $f(x) = x^2 + 4x + 36$ is increasing if $x \geq -2$.

HW Is $s = (t^2 - 3)^4$ an increasing function? In which range of the values of *t*?

HW Is $F = Ae^{rt}$ an increasing function? For which range of values of *t*? (*r* is a positive parameter)

2.4 Second- and Higher-Order Derivatives

Since the derivative $f'(x)$ is also a function of *x*, we can take the derivative of $f'(x)$, and this is called the second-order derivative of *f* at *x*.

Definition The ***second-order derivative*** of $f : \boldsymbol{S} \to \boldsymbol{R}$, $\boldsymbol{S} \subseteq \boldsymbol{R}$, is given by,

$$\begin{aligned} f''(x) &= \frac{d}{dx} f'(x) \\ &= \frac{d^2}{dx^2} f(x). \\ &= \frac{d^2 y}{dx^2}. \end{aligned}$$

- The k^{th}***-order derivative*** of $f : \boldsymbol{S} \to \boldsymbol{R}$, $\boldsymbol{S} \subseteq \boldsymbol{R}$, is recursively given by,

  $$f^k(x) = \frac{d}{dx} f^{k-1}(x),$$

  where $f^{k-1}(x)$ is the $(k-1)^{\text{st}}$ -order derivative of *f* at $x,\ k = 2,3,\dots$.
- *f* is twice-differentiable at *x* if $f''(x)$ exists.
- *f* is *n*-time-differentiable at *x* if $f^n(x)$ exists.
- *f* is twice-differentiable, write $f \in \boldsymbol{C}^2$, if it is twice-differentiable at each $x \in \boldsymbol{S}$.

• $f \in \boldsymbol{C}^k$ if it is *k*-time-differentiable at each $x \in \boldsymbol{S}$.

HW Find the second-order derivatives of the following functions.

a) $s = \left(t^2 - 3\right)^4$

b) $s = \boldsymbol{ln}\left(t^2 - 3\right)$

c) $z = \dfrac{3}{\left(1 - y^2\right)^{0.4}}$

d) $y = 1 + x + x^2 + \cdots + x^n$

e) $PV = A\left[1 + \dfrac{1}{1+r} + \dfrac{1}{(1+r)^2} + \cdots + \dfrac{1}{(1+r)^n}\right]$

f) $F = Ae^{rt}$.

2.5 Optimization: Single Variable

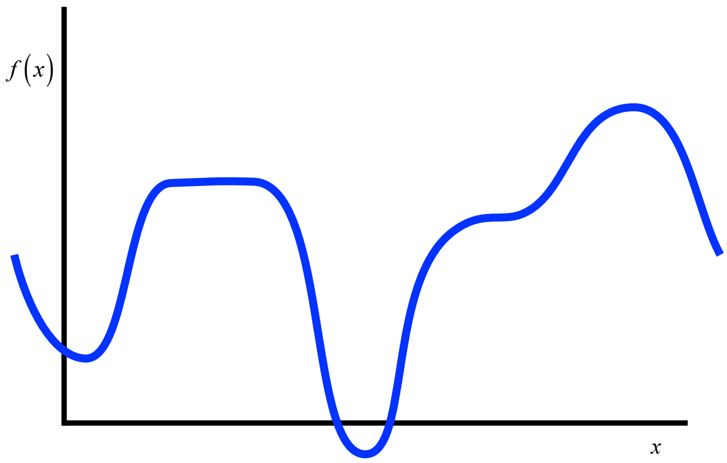

Figure 2.3 Graph of a function with maximum and minimum points.

Definition A point x^* is a ***local maximum point*** of $f\colon \boldsymbol{S} \to \boldsymbol{R}$, $\boldsymbol{S} \subseteq \boldsymbol{R}$, if

$$f\left(x^*\right) \geq f\left(x\right),$$

for any $x \in \boldsymbol{S}$ such that $\left|x - x^*\right| < \varepsilon$ for some $\varepsilon > 0$.

Definition A point x^* is a ***local strict maximum point*** of $f\colon \boldsymbol{S} \to \boldsymbol{R}$, $\boldsymbol{S} \subseteq \boldsymbol{R}$, if

$$f\left(x^*\right) > f\left(x\right),$$

for any $x \in \boldsymbol{S}$, $x \neq x^*$, such that $\left|x - x^*\right| < \varepsilon$ for some $\varepsilon > 0$.

Definition A point x^* is a ***global maximum point*** of $f\colon \boldsymbol{S} \to \boldsymbol{R}$, $\boldsymbol{S} \subseteq \boldsymbol{R}$, if

$$f\left(x^*\right) \geq f\left(x\right),$$

for any $x \in \boldsymbol{S}$.

Definition A point x^* is a ***global strict maximum point*** of $f\colon \boldsymbol{S} \to \boldsymbol{R}$, $\boldsymbol{S} \subseteq \boldsymbol{R}$, if

$$f\left(x^*\right) > f\left(x\right),$$

for any $x \in \boldsymbol{S}$, $x \neq x^*$.

- Definitions for minimum points are defined similarly.

HW Using the definition of the maximum and minimum points,

a) Show that x^* is a local maximum point of $f\left(x\right)$ if and only if it is a local minimum point of $-f\left(x\right)$. That is, $\boldsymbol{max}\, f\left(x\right) \Leftrightarrow \boldsymbol{min}\ -f\left(x\right)$.

b) Show that $\boldsymbol{max}\, f\left(x\right) \Leftrightarrow \boldsymbol{max}\ cf\left(x\right),\ c > 0$.

2.5.1 Necessary Conditions When a maximum point where the function is twice differentiable, two conditions follow.

Theorem If x^* is a ***local maximum point*** of $f\colon \boldsymbol{S} \to \boldsymbol{T}$, $\boldsymbol{S} \subseteq \boldsymbol{R}$, and $f \in \boldsymbol{C}^2$, then

1) $f'\left(x^*\right) = 0$, and

2) $f''(x^*) \le 0$.

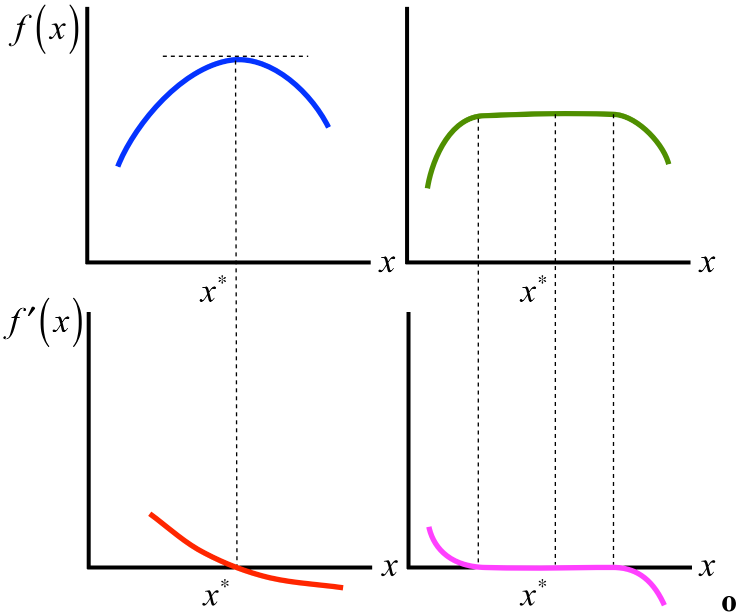

Figure 2.4 Graph of $f(x)$ and its derivative $f'(x)$ at the maximum points.

Example Profit Maximization.

$$\pi(q) = TR(q) - TC(q)$$
$$\frac{d\pi(q)}{dq} = \frac{dTR(q)}{dq} - \frac{dTC(q)}{dq}$$
$$= MR(q) - MC(q).$$

If q^* is an output level that maximizes (locally) profit, by the first order necessary condition,

$$\frac{d\pi(q^*)}{dq} = 0 \Leftrightarrow MR(q^*) = MC(q^*),$$

and second order necessary condition,

$$\frac{d^2\pi(q^*)}{dq^2} \le 0$$
$$\Leftrightarrow \frac{dMR(q^*)}{dq} \le \frac{dMC(q^*)}{dq}$$
$$\Leftrightarrow \text{slope of } MR(q^*) \le \text{ slope of } MC(q^*).$$

2.5.2 Sufficient Conditions What are the conditions that once attained are sufficient to announce a point being a maximum one. The necessary condition is not enough to guarantee a maximum point as seen in the graph below.

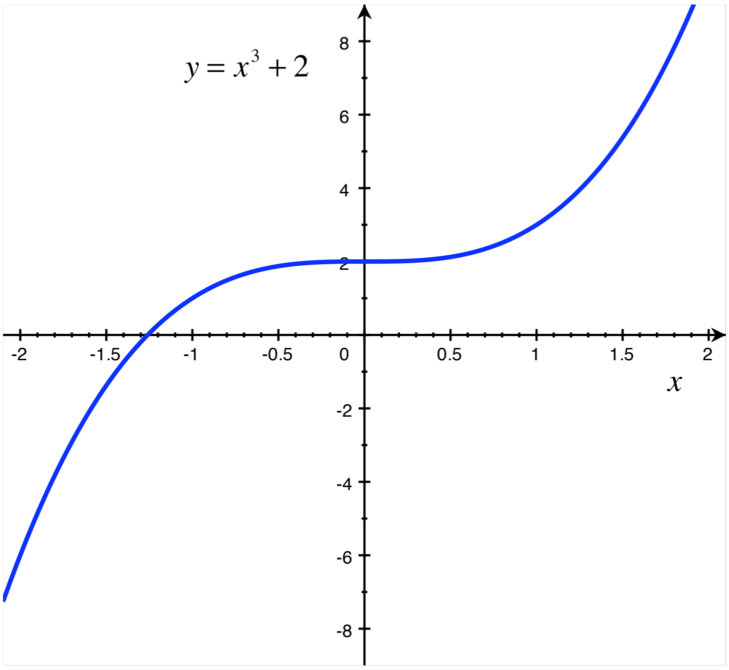

Figure 2.5 Graph of $y = x^3 + 2$.

Theorem If $f\colon \boldsymbol{S} \to \boldsymbol{R}$, $\boldsymbol{S} \subseteq \boldsymbol{R}$, $f \in \boldsymbol{C}^2$ and

1) $f'\left(x^*\right) = 0$, and

2) $f''\left(x^*\right) < 0$,

then x^* is a ***local maximum point***.

- Sufficient conditions actually guarantee local *strict* maximum
- Sufficient conditions cannot guarantee global maximum nor minimum. We need a much stronger assumption for global maximum.
- Each point x^* that satisfies the first-order sufficient condition is called a ***critical point***. Then only the critical point that also satisfies the second-order sufficient condition will be guaranteed to be a maximum point.
- There could be others maximum points that are not detected by the sufficient conditions.

Example Find the point of maximum profit. $TR(q)=10q^{\frac{1}{2}}$, $TC(q)=20+0.6q$.

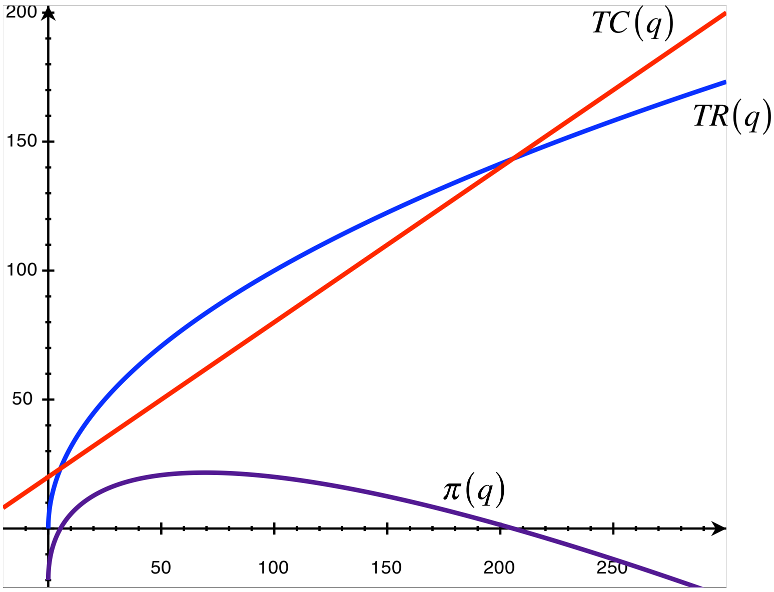

Figure 2.6 Profit maximization.

$$\pi(q)=10q^{\frac{1}{2}}-(20+0.6q)$$

$$\frac{d\pi(q^*)}{dq}=5(q^*)^{-\frac{1}{2}}-.6=0$$

$$q^*=\left(\frac{.6}{5}\right)^{-2}=69.44,$$

$$\text{and } \frac{d^2\pi(q^*)}{dq^2}=-\frac{5}{2}(q^*)^{-\frac{3}{2}}<0 .$$

We can conclude that the critical point $q^*=69.44$ is a local strict maximum point.

HW As a continuation of the previous example,

- Find the value of the profit at $q^*=69.44$.
- If $TR(q)=10q^{\frac{1}{2}}$ as before but $TC(q)=2000+0.6q$, will $q^*=69.44$ still be the point of local maximum profit?

HW Find the maximum and minimum points of the functions

a) $f(x) = (x^2 - 16)^3$.

b) $f(x) = x^3 + 6x^2 - 36x + 90$

c) $f(x) = x^2 e^x$

d) $f(x) = \dfrac{5x+2}{x^2+1}$

Theorem Let the composite function $h{:}\boldsymbol{S} \to \boldsymbol{U}$ be given by $h(x) = g(f(x))$, where $f{:}\boldsymbol{S} \to \boldsymbol{T}$ and $g{:}\boldsymbol{T} \to \boldsymbol{U}$. If the function $g{:}\boldsymbol{T} \to \boldsymbol{U}$ be an strictly increasing function such that its derivative is always positive, i.e., $g'(y) > 0$ for any $y \in \boldsymbol{T}$, then for any $x^* \in \boldsymbol{S}$

$$\left.\begin{aligned} f'(x^*) &= 0 \\ f''(x^*) &< 0 \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} h'(x^*) &= 0 \\ h''(x^*) &< 0. \end{aligned}\right.$$

Proof By the Chain Rule,

$$\begin{aligned} h'(x^*) &= g'(f(x^*))f'(x^*) = 0 \\ h''(x^*) &= g'(f(x^*))f''(x^*) + f'(x^*)\frac{d}{dx}g'(f(x^*)) \\ &= g'(f(x^*))f''(x^*) < 0. \end{aligned}$$

Example Consider the function $f(x) = 3 - x^2$. Using the sufficient conditions, we obtain a local maximum point $x^* = 0$. Then, this point x^* also satisfies the sufficient conditions for the functions

$$\begin{aligned} h(x) &= e^{f(x)} = e^{3-x^2} \\ r(x) &= \boldsymbol{ln}(f(x)) = \boldsymbol{ln}(3-x^2),\ 3 - x^2 > 0. \end{aligned}$$

But the point $x^* = 0$ does not satisfy the sufficient conditions for the function

$$s(x) = g(y) = (y-3)^2 = (-x^2)^2 = x^4 .$$

Can you verify this?

HW

1. Under the same assumptions as in the above theorem, is it true that

$$\left.\begin{array}{l} f'(x^*) = 0 \\ f''(x^*) \le 0 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} h'(x^*) = 0 \\ h''(x^*) \le 0. \end{array}\right.$$

2. If $g'(y) \ge 0$ for any $y \in \boldsymbol{T}$, does any conclusion of the previous theorem has to be modified?
3. If $g'(y) < 0$ for any $y \in \boldsymbol{T}$, does any conclusion of the previous theorem has to be modified?
4. Can we assume in the previous theorem only that $g'(y^*) > 0$ only at $y^* = f(x^*)$?

2.6 Concave and Convex Functions

Definition The function $f : \boldsymbol{S} \to \boldsymbol{T}$, $\boldsymbol{S} \subseteq \boldsymbol{R}$, is a ***concave (convex)*** function, if

$$f(\lambda x_1 + (1-\lambda)x_2) \underset{(\le)}{\ge} \lambda f(x_1) + (1-\lambda) f(x_2),$$

for any $x_1, x_2 \in \boldsymbol{S}$, and $0 \le \lambda \le 1$.

Definition The function $f : \boldsymbol{S} \to \boldsymbol{R}$, $\boldsymbol{S} \subseteq \boldsymbol{R}$, is a ***strictly concave (convex)*** function, if

$$f(\lambda x_1 + (1-\lambda)x_2) \underset{(<)}{>} \lambda f(x_1) + (1-\lambda) f(x_2),$$

for any $x_1, x_2 \in \boldsymbol{S}$, and $0 < \lambda < 1$.

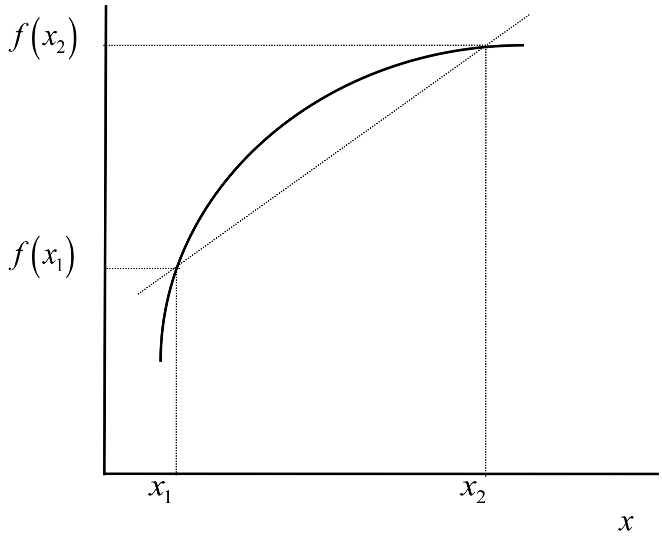

Figure 2.7 A concave function.

- According to the definition, a function is concave if the straight line connecting any pair of points on the graph of the function is never above the graph of the function.
- Linear function is both concave and convex.
- If *f* and *g* are concave, then $f+g$ is also concave.
- If *f* is concave, then $-f$ is convex.

HW If f and g are concave, then is $f-g$ also concave?

Theorem If *f* is twice differentiable, then

a) *f* is concave $\Leftrightarrow f''(x) \le 0$, for any $x \in \boldsymbol{S}$.

b) *f* is strictly concave $\Leftrightarrow f''(x) < 0$, for any $x \in \boldsymbol{S}$.

HW Which of the following functions concave or convex?

a). $f(x) = x^2$

b) $f(x) = x^2 - 6x + 5$

c) $f(x) = x^3 - 6x^2 + 5x + 100$

d) $f(x) = \boldsymbol{ln}\, x$

Theorem

a)If f is concave, a local maximum point x^* is also a global one.

b)If f is strictly concave, a local maximum point x^* is also a strictly global one.

Theorem

a)If f is strictly concave, whenever $f'(x^*)=0$, x^* is a strictly global maximum point.

b)If f is concave, whenever $f'(x^*)=0$, x^* is a global maximum point.

HW (Continued) Are the maximum and minimum points of the functions below global ones?

a) $f(x)=(x^2-16)^3$.

b) $f(x)=x^3+6x^2-36x+90$

c) $f(x)=x^2e^x$

d) $f(x)=\dfrac{5x+2}{x^2+1}$

Additional HW (optional)**:** Baldani, et. al. [2005], p. 36, #A3, A4, A6.

2.7 Differentials

The derivative $\dfrac{dy}{dx}$, $y=f(x)$, can be used to approximate a change in y, Δy, as the result of a change in x, Δx. That is,

$$f(x_0+\Delta x)\approx f(x_0)+f'(x_0)\Delta x,$$

and thus the change in y is approximated by,

$$\Delta y = f(x_0+\Delta x)-f(x_0)\approx f'(x_0)\Delta x .$$

From the Figure below, the smaller the value of Δx the better the approximation of Δy.

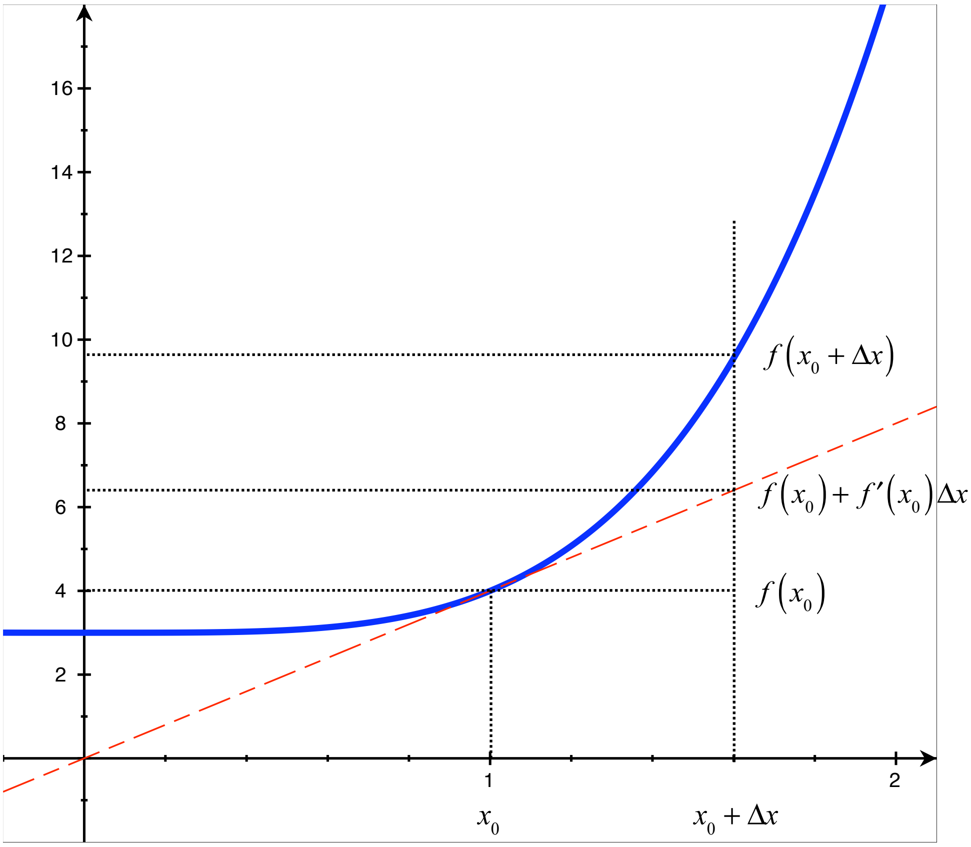

Figure 2.8 The approximation of the change in $y = f(x)$ when x changes.

See Taylor's Polynomial for a more general approximation.

We define the differential of y at x^* as this approximation when the change in x is approaching zero.

Definition Let $f : \boldsymbol{S} \to \boldsymbol{R}$, , be a function, and $y = f(x)$. The ***differential*** of y at x^*, denoted by dy, is given by,

$$dy = f'(x_0)\,dx,$$

where dx is just Δx as it is approaching zero.

- Whenever we have derivative, we have differential, and vice versa.

Example From the Total Revenue and Total Cost functions $TR(q)=10q^{\frac{1}{2}}$ and $TC(q)=20+0.6q$, the differentials are given by

$$dTR(q)=MR(q)dq=5q^{-\frac{1}{2}}dq$$
$$dTC(q)=MC(q)dq=0.6dq.$$

At $q^*=64$, if one more unit of output is produced, then $\Delta q=1$ and the *additional revenue received* is

$$\Delta TR \approx 5(64)^{-\frac{1}{2}}\Delta q=\frac{5}{8}\Delta q=\frac{5}{8},$$

and *additional cost incurred* is

$$\Delta TC=0.6\Delta q=0.6.$$

Note that the additional cost is exactly 0.6 because the cost function is linear.

The followings are formulae of derivative frequently used. Let $y=f(x)$ and $z=g(x)$.

a) $d(Ay+Bz)=Ady+Bdz$, A and B are constants.
b) $d(yz)=zdy+ydz$.
c) $d\left(\frac{y}{z}\right)=\frac{zdy-ydz}{z^2}$.
d) Chain Rule: If $w=h(y)$, then $dw=h'(y)dy=h'(y)f'(x)dx$.

Note that dy and dz involve $f'(x)$ and $g'(x)$ and the derivative formulae in Section 2.1 apply.

HW (Continued) Find the differential of the profit function. At $q^*=64$, what is the approximate additional profit if we produce one more unit?

HW Wrtie the differential of the following functions.

a) $s = \left(t^2 - 3\right)^4$

b) $s = \boldsymbol{ln}\left(t^2 - 3\right)$

c) $z = \dfrac{3}{\left(1 - y^2\right)^{0.4}}$

d) $y = 1 + x + x^2 + \cdots + x^n$

e) $PV = A\left[1 + \dfrac{1}{1+r} + \dfrac{1}{(1+r)^2} + \cdots + \dfrac{1}{(1+r)^n}\right]$

f) $F = Ae^{rt}$.