

Chapter 7 Functions of Several Variables

7.1 Functions of Two or more Variables

Notation and Terminology

There are familiar formulas in which a given variable depends on two or more other variables. For example,

- Area A of a triangle depends on the base length b and height h by the formula $A = \frac{1}{2}bh$.
- The volume V of a rectangular box depends on the length l , the width w , and the height h by the formula $V = lwh$.
- The arithmetic average $\bar{x}$ of n real numbers, $x_1, x_2, \dots, x_n$, depends on those numbers by the formula $\bar{x} = \frac{1}{n}(x_1 + x_2 + \dots + x_n)$.

We say that

- A is a function of the two variables b and h .
- V is a function of the three variables l , w , and h
- $\bar{x}$ is a function of the n variables $x_1, x_2, \dots, x_n$.

The terminology and notation for functions of two or more variables is similar to that for functions of one variable. For example,

$$z = f(x, y)$$

means that z is a function of x and y in the sense that unique value of the dependent variable z is determined by specifying values for the independent variables x and y .

7.1.1 DEFINITION A **function f of two variables**, x and y , is a rule that assigns a unique real number $f(x, y)$ to each point (x, y) in some set D , the domain of f , in the xy -plane.

7.1.2 DEFINITION A **function f of three variables**, x , y , and z , is a rule that assigns a unique real number $f(x, y, z)$ to each point (x, y, z) in some set D , the domain of f , in three dimensional space.

Example 1: Let $f(x, y) = \sqrt{y+1} + \ln(x^2 - y)$. Find $f(e, 0)$ and sketch the natural domain of f .

Example 2: Let $f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$. Find $f\left(0, \frac{1}{2}, -\frac{1}{2}\right)$.

7.2 Partial Derivatives

7.2.1 Partial Derivatives of Functions of Two Variables

Definition: If $z = f(x, y)$ and (x_0, y_0) is a point in the domain of f , then the **partial derivative of f with respect to x** at (x_0, y_0) [also called the **partial derivative of z with respect to x** at (x_0, y_0)] is the derivative at x_0 of the function that results when $y = y_0$ is held fixed and x is allowed to vary. This partial derivative is denoted by $f_x(x_0, y_0)$ and is given by

$$f_x(x_0, y_0) = \left. \frac{d}{dx} [f(x, y_0)] \right|_{x=x_0}$$

Similarly, the **partial derivative of f with respect to y** at (x_0, y_0) [also called the **partial derivative of z with respect to y** at (x_0, y_0)] is the derivative at y_0 of the function that results when $x = x_0$ is held fixed and y is allowed to vary. This partial derivative is denoted by $f_y(x_0, y_0)$ and is given by

$$f_y(x_0, y_0) = \left. \frac{d}{dy} [f(x_0, y)] \right|_{y=y_0}$$

Notations for Partial Derivatives

If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = D_y f$$

Example 1: Find $f_x(1, 3)$ and $f_y(1, 3)$ for the function $f(x, y) = 2x^3y^2 + 2y + 4x$.

Example 2: If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

Example 3: Find $\partial z / \partial x$ and $\partial z / \partial y$ if $z = x^4 \sin(xy^3)$.

Example 4: If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $\partial f / \partial x$ and $\partial f / \partial y$

7.2.2 Partial Derivatives Viewed as Rates of Change and Slopes

Recall that if $y = f(x)$, then the value of $f'(x_0)$ can be interpreted either as the rate of change of y with respect to x at x_0 or as the slope of the tangent line to the graph of f at x_0 . Partial derivatives have analogous interpretations.

Example 5: The wind chill temperature index is given by the formula

$$W = 35.74 + 0.6215T + (0.4275T - 35.75)v^{0.16}$$

Compute the partial derivative of W with respect to v at the point $(T, v) = (25, 10)$ and interpret this partial derivative as a rate of change.

7.2.3 Implicit Partial Differentiation

Example 8: Find the slope of the sphere $x^2 + y^2 + z^2 = 1$ in the y -direction at the points

$$\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right) \text{ and } \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right).$$

Example 9: Suppose that $D = \sqrt{x^2 + y^2}$ is the length of the diagonal of a rectangle whose sides have lengths x and y that are allowed to vary. Find a formula for the rate of change of D with respect to x if x varies with y held constant, and use this formula to find the rate of change of D with respect to x at the point where $x = 3$ and $y = 4$.

Example 10: Find $\partial z / \partial x$ and $\partial z / \partial y$ if z is defined implicitly as a function of x and y by the equation

$$x^3 + y^3 + z^3 + 6xyz = 1$$

7.2.4 Partial Derivatives of Functions with more than Two Variables

For a function $f(x, y, z)$ of three variables, there are three partial derivatives:

$$f_x(x, y, z), \quad f_y(x, y, z), \quad f_z(x, y, z)$$

If a dependent variable

$$w = f(x, y, z)$$

is used, then the three partial derivatives of f can be denoted by

$$\frac{\partial w}{\partial x}, \quad \frac{\partial w}{\partial y}, \quad \frac{\partial w}{\partial z}$$

Example 11: If $f(x, y, z) = x^3 y^2 z^4 + 2xy + z$. Find $f_x(x, y, z)$, $f_y(x, y, z)$, $f_z(x, y, z)$, and $f_z(-1, 1, 2)$.

Example 12: Find f_x , f_y , and f_z if $f(x, y, z) = e^{xy} \ln z$

7.2.5 Higher-Order Partial Derivatives

Suppose that f is a function of two variable x and y . Since the partial derivatives $\partial f / \partial x$ and $\partial f / \partial y$ are also functions of x and y , these functions may themselves have partial derivatives. This gives rise to four possible **second-order** partial derivatives of f , which are defined by

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = f_{xx} \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = f_{yy}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = f_{xy} \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = f_{yx}$$

The last two cases are called the ***mixed second-order partial derivatives*** or the ***mixed second partials***.

Example 13: Let $f(x, y) = y^2 e^x + y$. Find $\frac{\partial^3 f}{\partial y^2 \partial x}$.

Example 14: Find the second partial derivatives of

$$f(x, y) = x^3 + x^2y^3 - 2y^2$$

Notice that $f_{xy} = f_{yx}$ in Example 14. This is not just a coincidence. It turns out that the mixed partial derivatives f_{xy} and f_{yx} are equal for most functions that one meets in practices.

7.3 The Chain Rule

7.3.1 Chain Rules for Derivatives

Theorem (Chain Rules for Derivatives): If $x = x(t)$ and $y = y(t)$ are differentiable at t , and if $z = f(x, y)$ is differentiable at the point $(x, y) = (x(t), y(t))$, then $z = f(x(t), y(t))$ is differentiable at t and

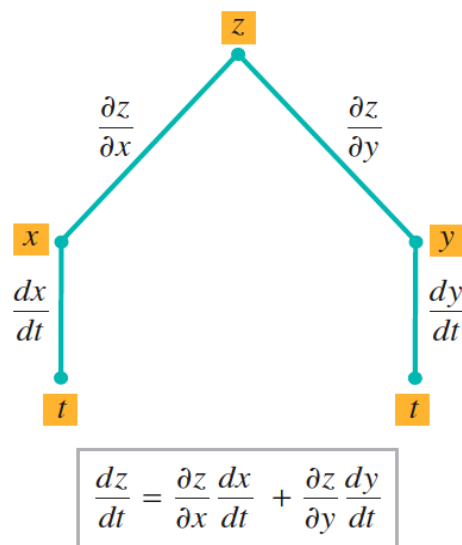
$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y) .

If each of the functions $x = x(t)$, $y = y(t)$, and $z = z(t)$ is differentiable at t , and if $w = f(x, y, z)$ is differentiable at the point $(x, y, z) = (x(t), y(t), z(t))$, then the function $w = f(x(t), y(t), z(t))$ is differentiable at t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y, z) .



Example 1: Suppose that

$$z = x^2y, \quad x = t^2, \quad y = t^3$$

Use the chain rule to find dz/dt , and check the result by expressing z as a function of t and differentiating directly.

Example 2: If

$$z = x^2y + 3xy^4 \quad \text{where} \quad x = \sin 2t, \quad y = \cos t$$

Find dz/dt when $t = 0$.

Example 3: Suppose that

$$w = \sqrt{x^2 + y^2 + z^2}, \quad x = \cos \theta, \quad y = \sin \theta, \quad z = \tan \theta$$

Use the chain rule to find $dw/d\theta$ when $\theta = \pi/4$

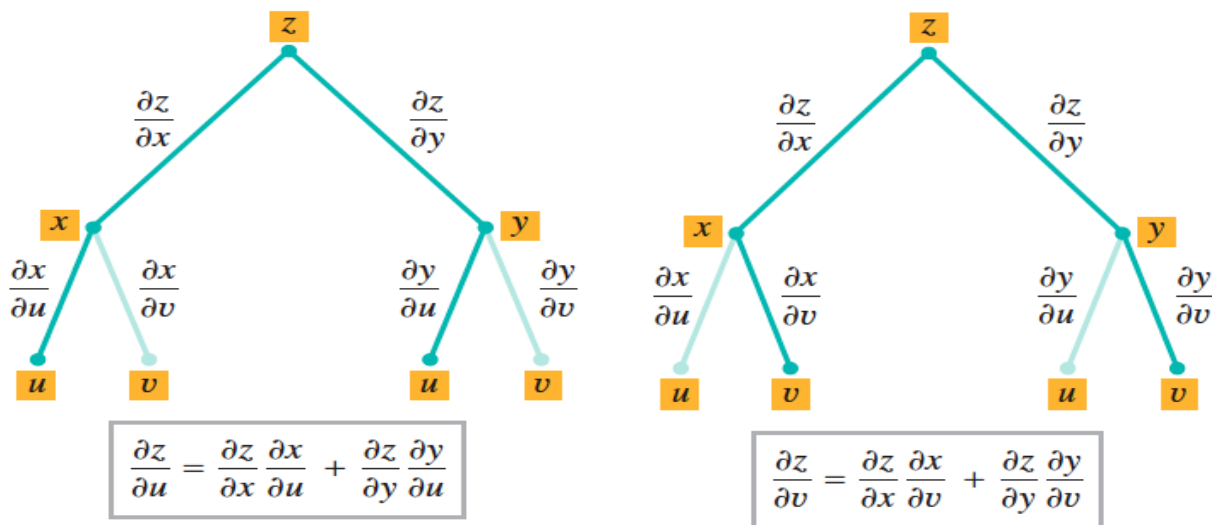
7.3.2 Chain Rules for Partial Derivatives

Theorem (Chain Rules for Partial Derivatives): If $x = x(u, v)$ and $y = y(u, v)$ have first-order partial derivatives at the point (u, v) , and if $z = f(x, y)$ is differentiable at the point $(x, y) = (x(u, v), y(u, v))$, then $z = f(x(u, v), y(u, v))$ has first order partial derivatives at the point (u, v) given by

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}$$

If each function $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$ has first-order partial derivatives at the point (u, v) , and if $w = f(x, y, z)$ is differentiable at the point $(x, y, z) = (x(u, v), y(u, v), z(u, v))$, then $w = f(x(u, v), y(u, v), z(u, v))$ has first order partial derivatives at the point (u, v) given by

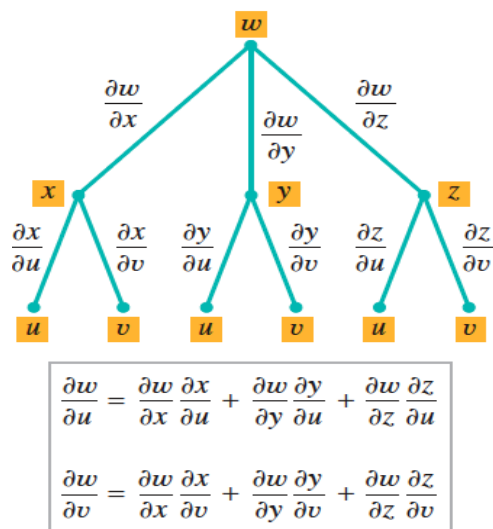
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u} \quad \text{and} \quad \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v}$$



Example 5: Given that

$$z = e^{xy}, \quad x = 2u + v, \quad y = u/v$$

Find $\partial z / \partial u$ and $\partial z / \partial v$ using the chain rule.



Example 6: Suppose that

$$w = e^{xyz}, \quad x = 3u + v, \quad y = 3u - v, \quad z = u^2v$$

Find $\partial w / \partial u$ and $\partial w / \partial v$ using the chain rule.

Example 7: Suppose that $w = x^2 + y^2 - z^2$ and

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi$$

Use appropriate forms of the chain rule to find $\partial w / \partial \rho$ and $\partial w / \partial \theta$

Example 8: Suppose that

$$w = xy + yz, \quad y = \sin x, \quad z = e^x$$

Use an appropriate form of the chain rule to find dw / dx

7.3.3 Implicit Differentiation

Consider the special case where $z = f(x, y)$ is a function of x and y is a differentiable function of x . Equation (5) then becomes

$$\frac{dz}{dx} = \frac{\partial f}{\partial x} \frac{dx}{dx} + \frac{\partial f}{\partial y} \frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} \quad (*)$$

This result can be used to find derivatives of functions that are defined implicitly. For example, suppose that the equation

$$f(x, y) = c \quad (**)$$

Defines y implicitly as a differentiable function of x and we are interested in finding dy/dx . Differentiating both sides of (**) with respect to x and applying (*) yields

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} = 0$$

Thus, if $\partial f / \partial y \neq 0$, we obtain

$$\frac{dy}{dx} = - \frac{\partial f / \partial x}{\partial f / \partial y}$$

In summary, we have the following result.

Theorem: If the equation $f(x, y) = c$ defines y implicitly as a differentiable function of x , and if $\frac{\partial f}{\partial y} \neq 0$, then

$$\frac{dy}{dx} = - \frac{\partial f / \partial x}{\partial f / \partial y}$$

Example 10: Given that $x^3 + y^2x - 3 = 0$, find dy/dx using the Theorem and check the result using implicit differentiation.

Example 11: Find dy/dx if $x^3 + y^3 = 6xy$ using the Theorem.

The chain rule also applies to implicit partial differentiation. Consider the case where $w = f(x, y, z)$ is a function of x, y , and z and z is a differentiable function of x and y . It follows from Theorem that

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial x} \quad (*)$$

If the equation

$$f(x, y, z) = c \quad (**)$$

defines z implicitly as a differentiable function of x and y , then taking the partial derivative of each side of $(**)$ with respect to x and applying $(*)$ gives

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial x} = 0$$

If $\partial f / \partial z \neq 0$, then

$$\frac{\partial z}{\partial x} = -\frac{\partial f / \partial x}{\partial f / \partial z}$$

A similar result holds for $\partial z / \partial y$.

Theorem: If the equation $f(x, y, z) = c$ defines z implicitly as a differentiable function of x and y , and if $\frac{\partial f}{\partial z} \neq 0$, then

$$\frac{\partial z}{\partial x} = -\frac{\partial f / \partial x}{\partial f / \partial z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\partial f / \partial y}{\partial f / \partial z}$$

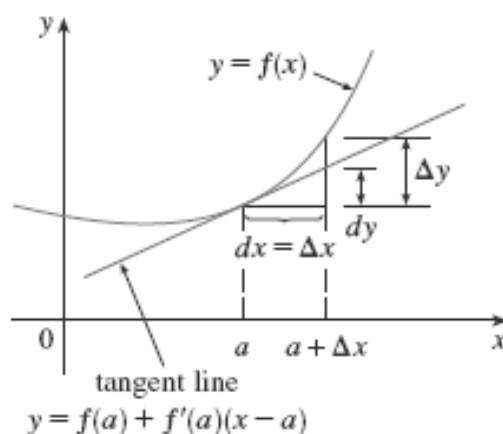
Example 12: Consider the sphere $x^2 + y^2 + z^2 = 1$. Find $\partial z / \partial x$ and $\partial z / \partial y$ at the point

$$\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$$

7.4 Total Differential and Its Applications

For a differentiable function of one variable, $y = f(x)$, we define the differential dx to be an independent variable; that is, dx can be given the value of any real number. The differential of y is then defined as

$$dy = f'(x)dx$$



The figure above shows the relationship between the increment Δy and the differential dy : Δy represents the change in height of the curve $y = f(x)$ and dy represents the change in height of the tangent line when x changes by an amount $dx = \Delta x$.

For a differentiable function of two variables, $z = f(x, y)$, we define the differentials dx and dy to be independent variables; that is, they can be given any values. Then the **differential** dz , also called the **total differential**, is defined by

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \quad (*)$$

Sometimes the notation df is used in place of dz .

If we take $dx = \Delta x = x - a$ and $dy = \Delta y = y - b$ in Equation (*), then the differential of z is

$$dz = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

So, in the notation of differentials, the **linear approximation**

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

can be written as

$$f(x,y) \approx f(a,b) + dz$$

Note: The **linearization** of f at (a,b) is

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Example 1:

- (a) If $z = f(x,y) = x^2 + 3xy - y^2$, find the differential dz .
- (b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz .

Example 2: Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.