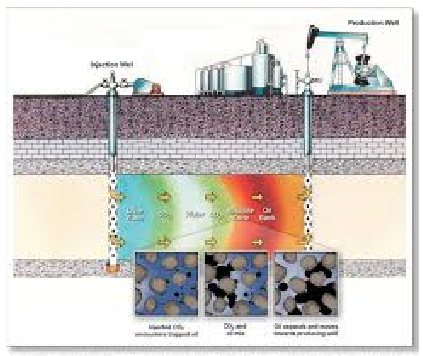
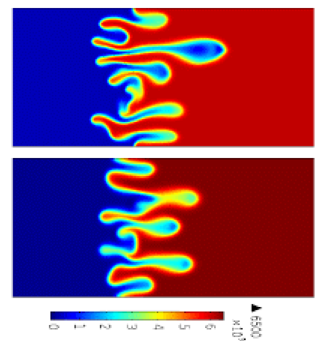
Chapter 1: Systems of Linear Equations and Matrices

1 Introduction/Motivation

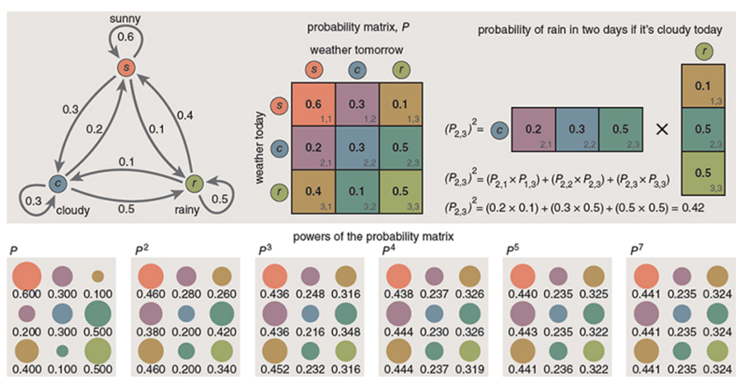
Linear algebra has been used as the fundamental concept in many important applications, such as:

- Solving certain systems of differential equations.



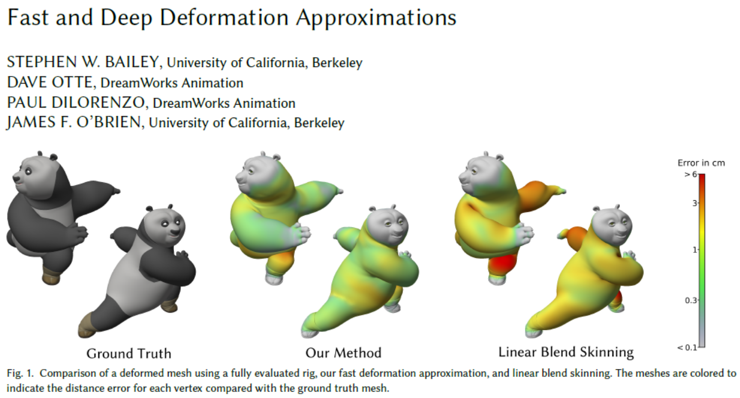


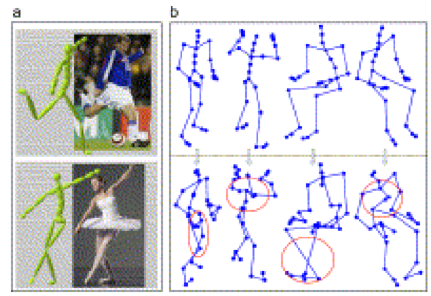
- Markov chains as models of random processes.



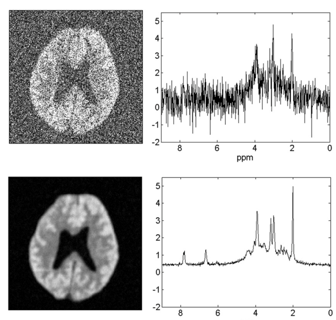
- The Google PageRank algorithm for search engines.

- Animation (Linear Transformation)





- Image processing (SVD,Denosing,foreground-background separation, Low-rank approximation)



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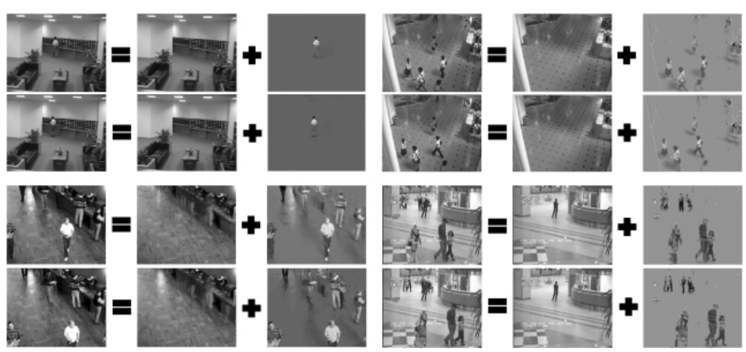


2 Notations and Review of Matrix Algebra

2.1 Notations

- $\mathbb{R}$ is the set of all real numbers (scalars).
- $\mathbb{R}^n$ is the set of column vectors with n entries E.g.
- $\mathbb{R}^{m\times n}$ or $M_{m\times n}(\mathbb{R})$ is the set of all $m \times n$ matrices (with m rows and n columns, i.e. height m and width n). E.g. $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & -1 \\ 2 & 4 & 7 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 4}$.
- $M_n(\mathbb{R}) = \mathbb{R}^{n\times n}$ or $M_{n\times n}(\mathbb{R})$ is the set of all $n \times n$ matrices. E.g. $\begin{bmatrix} 1 & 2 & 3 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 1 & 0 & 0 & 1 \end{bmatrix}$.
- $\mathbf{I}_n$ is the $n \times n$ identity matrix. E.g. $\mathbf{I_4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

2.2 Transpose

Definition 2.1. The transpose of an $m \times n$ matrix $\mathbf{A}$ is the $n \times m$ matrix $\mathbf{A}^T$ obtained by flipping $\mathbf{A}$ over, so the (i, j)th entry in $\mathbf{A}^T$ is the same as the (j, i)th entry in A.

Example 2.1. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & -1 \\ 2 & 4 & 7 & 1 \end{bmatrix}$. Find $\mathbf{A}^T$.

2.3 Dot product

For column vectors $u, v \in \mathbb{R}^n$, the **dot product** is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \ldots u_n v_n = \sum_{i=1}^{n} u_i v_i$$

Example 2.2. Find $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 10 \\ 20 \\ 30 \\ 40 \end{bmatrix}$.

2.4 Product of a matrix and a vector

We can multiply an $m \times n$ matrix by a vector in $\mathbb{R}^n$ to get a vector in $\mathbb{R}^m$, for example

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} =$$

3 Introduction to Systems of Linear Equations

3.1 Systems of Linear Equations

We consider a linear equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $a_1, a_2, \ldots, a_n$ are *constants* and $x_1, x_2, \ldots, x_n$ are *variables* or *unknowns*. Notice, this is *linear* in term of the *power* of x_i, $i = 1, \ldots, n$.

- The variables $x_1, x_2, \ldots, x_n$ that solve or satisfy the equation is called **solution** of that equation.
- This equation generally has infinitely many solutions when $n \geq 2$.

System of linear equations
The general form of system of linear equations with m equations and n unknowns is:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad &\vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

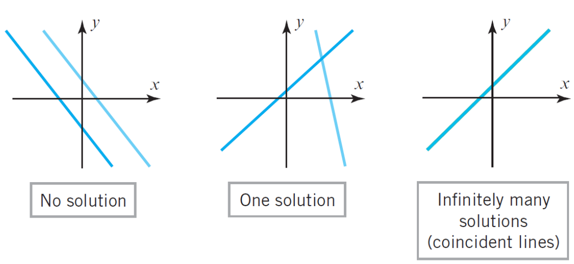
where a_{ij} and b_i are constant real numbers, for for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$; $x_1, \ldots, x_n$ are variables or unknowns.

For $m = n$, there are 3 possible cases for the solution of a system of linear equations with n equations and n unknowns (there are no other possibilities).

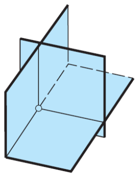
- There is no solution.
- There is exactly one unique solution.
- There are many solutions.

Definition 3.1. A linear system is **consistent** if it has at least one solution and **inconsistent** if it has no solutions.

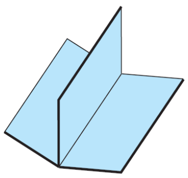
m = n = 2:



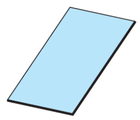
m = n=3



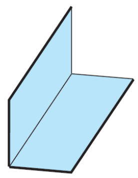
One solution
(intersection is a point)



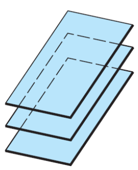
Infinitely many solutions
(intersection is a line)



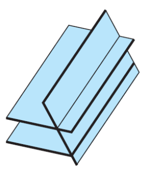
Infinitely many solutions
(planes are all coincident;
intersection is a plane)



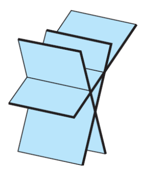
Infinitely many solutions
(two coincident planes;
intersection is a line)



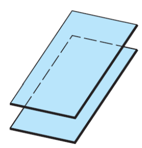
No solutions
(three parallel planes;
no common intersection)



No solutions
(two parallel planes;
no common intersection)



No solutions
(no common intersection)



No solutions
(two coincident planes
parallel to the third;
no common intersection)

- The general form of system of linear equations is:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad & \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

  where a_{ij} and b_i are some constant numbers, for $i, j = 1, 2, \ldots, n$.

- This system can be written in the matrix-vector form as:

$$\mathbf{Ax} = \mathbf{b} \quad \text{where} \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

- **Augmented matrix** of the above linear system is given by

$$[A|b] \quad \text{or} \quad \left[\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right] \quad \text{or} \quad \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{bmatrix}.$$

We can solve the system of linear equations by using the methods called, **Gaussian elimination** and **Gauss-Jordan elimination**.

Main concept: Change the *augmented matrix* to an equivalent system that is easy to be solved, called **"reduced row-echelon form (RREF)"** by using the process called **"elementary row operation."**

4 Reduced Row Echelon Form (RREF)

Definition 4.1. Let $\mathbf{A}$ be a matrix of real numbers. Recall that $\mathbf{A}$ is said to be in *reduced row echelon form (RREF)* if the following hold:

RREF0: Any rows of zeros come at the bottom of the matrix, after all the nonzero rows.

RREF1: In any nonzero row, the first nonzero entry is equal to 1.

Each of these entries is called a **pivot** or a **leading 1**.

RREF2: In any nonzero row, the pivot is further to the right than the pivots in all previous rows.

RREF3: If a column contains a **pivot** or a **leading 1**, then all other entries in that column are zero.

Note: A matrix that has the first three properties is said to be in **row echelon form (REF)** . (Thus, a matrix in reduced row echelon form is of necessity in row echelon form, but not conversely.)

Example 4.1. Consider the matrices.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $\mathbf{A}$ is not in RREF because the middle row is zero and the bottom row is not, so condition RREF0 is violated. The matrix
- $\mathbf{B}$ is also not in RREF because the first nonzero entry in the top row is 2 rather than 1, which violates RREF1.
- $\mathbf{C}$ is not in RREF because the pivot in the bottom row is to the left of the pivots in the previous rows, violating RREF2.
- The matrix $\mathbf{D}$ is not in RREF because the last column contains a pivot and also another nonzero entry, violating RREF3.

Example 4.2. Matrix $\mathbf{E} = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & 5 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is RREF

Example 4.3. .

The following matrices are in reduced row echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The following matrices are in row echelon form but not reduced row echelon form.

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 4.4. All matrices of the following types are in row echelon form (REF) (but may not RREF):

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

All matrices of the following types are in reduced row echelon form (RREF):

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Linear systems whose augmented matrix is in RREF or REF are generally easy to obtain the solutions as shown in the following examples.

Example 4.5. Suppose that the augmented matrix for a linear system in the unknowns x_1, x_2, x_3, and x_4 has been reduced by elementary row operations to the following reduced row echelon form $\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$. Find the solution of this linear system.

Example 4.6. Suppose that the augmented matrix for a linear system in the unknowns x, y, and z has been reduced by elementary row operations to the given reduced row echelon form. Solve the system.

(a) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(b) $\mathbf{B} = \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(c) $\mathbf{C} = \begin{bmatrix} 1 & 5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

If a linear system has infinitely many solutions, then a set of parametric equations from which all solutions can be obtained by assigning numerical values to the parameters is called a **general solution** of the system.

5 Elementary row operation

We now consider the following operations that does not change the solution of a linear system of equations.
- Interchange two equations.
- Multiply an equation through by a nonzero constant.
- Add a constant times one equation to another.
Since the rows (horizontal lines) of an augmented matrix correspond to the equations in the associated system, these three operations correspond to the following operations, called " **elementary row operations**" on the rows of the augmented matrix.

Definition 5.1. Let A be a matrix. The following operations on A are called **elementary row operations**:

ERO1: Exchange two rows.

ERO2: Multiply a row by a nonzero constant.

ERO3: Add a multiple of one row to another row.

Theorem 5.1. Let **A** be a matrix.

(a) By applying a sequence of row operations to **A**, one can obtain a matrix B that is in RREF.

(b) Although there are various different sequences that reduce A to RREF, they all give the same matrix B at the end of the process.

6 Solving Linear Systems

We will consider 2 approaches for solving systems of linear equations.

- **Gauss-Jordan elimination**
  - It reduces an augmented matrix to **reduced row echelon form (RREF)** by using the **elementary row operations**.
  - Zeros are introduced below and above the leading 1s.
  - Solve for solution(s) (possibly in terms of free variables).
- **Gaussian elimination**:
  - It reduces an augmented matrix to **row echelon form (REF)** by using the **elementary row operations**.
  - Zeros are introduced only below the leading 1s.
  - Use **Back substitution** to solve for solution(s) (possibly in terms of free variables).

Example 6.1. Find the solution of the following linear system

$$\begin{aligned} -2x_3 + 7x_5 &= 12 \\ 2x_1 + 4x_2 - 10x_3 + 6x_4 + 12x_5 &= 28 \\ 2x_1 + 4x_2 - 5x_3 + 6x_4 - 5x_5 &= -1 \end{aligned}$$

by using
(a) Gaussian elimination
(b) Gaussian-Jordan elimination

(continued)

Example 6.2. Solve the following linear system by Gauss-Jordan elimination.

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\ 5x_3 + 10x_4 + 15x_6 &= 5 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 6 \end{aligned}$$

7 Homogeneous Linear Systems

A system of linear equations with m equations and n unknowns is said to be **homogeneous** if the constant terms are all zero; that is, the system has the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots \quad & \vdots \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= 0 \end{aligned}$$

where a_{ij} and b_i are constant real numbers, for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$; $x_1, \ldots, x_n$ are variables or unknowns.

Remarks:

- Every homogeneous system of linear equations is **consistent** because all such systems have $x_1 = 0, x_2 = 0, ..., x_n = 0$ as a solution.
- This zero solution is called the **trivial solution**; if there are other solutions, they are called **nontrivial solutions**.
- There are only two possibilities for the solutions of homogeneous linear systems:
  - The system has only the trivial solution.
  - The system has infinitely many solutions in addition to the trivial solution.
- There is one case in which a homogeneous system is assured of having **nontrivial solutions**– namely, whenever the system involves **more unknowns than equations.**

Theorem 7.1. A homogeneous linear system with **more unknowns than equations** has **infinitely many solutions**.

- Elementary row operations do not alter columns of zeros in a matrix, so the reduced row echelon form of the augmented matrix for a homogeneous linear system has a final column of zeros. This implies that the linear system corresponding to the reduced row echelon form is homogeneous, just like the original system.
- The linear system corresponding to that reduced row echelon form will either have the same number of equations as the original system or it will have fewer —depending on whether or not the reduced row echelon form of the augmented matrix for a homogeneous linear system has any rows of zero. E.g. when we have rows of zero, it is corresponding to the equation

$$0 \cdot x_1 + 0 \cdot x_2 + \cdots + 0 \cdot x_n = 0$$

which can be ignored.

Theorem 7.2 (Free Variable Theorem for Homogeneous Systems)**.** If a homogeneous linear system has n unknowns, and if the **reduced row echelon form** of its augmented matrix has r nonzero rows, then the system has $n - r$ free variables.

Example 7.1. Use Gauss-Jordan elimination to solve the homogeneous linear system

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\ 5x_3 + 10x_4 + 15x_6 &= 0 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 0 \end{aligned}$$

Solution:
The augmented matrix for the given homogeneous system is:

The reduced row echelon form is given by (exercise!):

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system of equations is