Limit of a Function

- Limit- An informal Approach
- Limit Theorems
- Continuity
- Trigonometric Limits
- Limits that Involve Infinity
- Limit- a formal approach

1 Limit: Informal Approach

The following definition is an informal definition of limit.

Definition 1.1 (Informal/Working definition). The limit of $f(x)$ is L as x approaches a and write this as

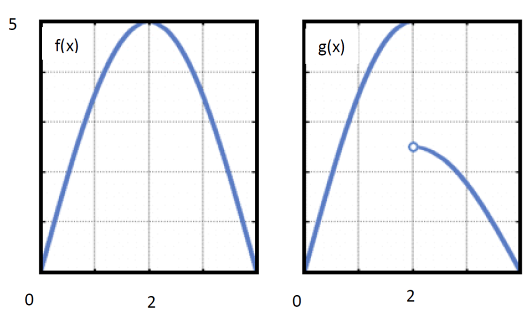
$$\lim_{x\to a} f(x) = L \tag{1}$$

provided we can make $f(x)$ as close to L as we want for all x sufficiently close to a, from both sides, without actually letting x be a.

An alternative notation for denoting $\lim_{x\to a} f(x) = L$ is

$$f(x) \to L \quad \text{as} \quad x \to a.$$

Example 1.1. Consider the following plots of $f(x)$ and $g(x)$.
What is $\lim_{x\to 2} f(x)$? What about $\lim_{x\to 2} g(x)$



Example 1.2. Suppose $f(x)$ is a function such that $\lim_{x \to a} f(x) = L$ where a and L are real numbers. Sketch a possible graph of $y = f(x)$ for each of the following properties of $f(a)$.

(a) $f(a) = L$.

(b) $f(a) = k \neq L$, where k is a real number.

(c) $f(a)$ is undefined.

Example 1.3. Consider the function

$$f(x) = \frac{x^2 - 4}{x - 2}$$

whose domain is $\mathcal{D} := \mathbb{R} - \{2\}$ or the set of all real numbers except 2. Notice that we cannot evaluate the function at $x = 1$ since it will be undefined with zero denominator. However, $f(x)$ can be evaluated at the values close to $x = 2$ as follows.

x	$f(x) = \frac{x^2-4}{x-2}$	x	$f(x) = \frac{x^2-4}{x-2}$
2.10000000	4.10000000	1.90000000	3.90000000
2.01000000	4.01000000	1.99000000	3.99000000
2.00100000	4.00099999	1.99900000	3.99900000
2.00010000	4.00010000	1.99990000	3.99990000
2.00001000	4.00001000	1.99999000	3.99999000

Table 1: Values of function $f(x) = \frac{x^2-4}{x-2}$ as x approaching 2 from the right and left directions.

- Notice from above that, although we cannot evaluate $f(x)$ at $x = 2$, the function value is getting close to 4 when the value of x is getting close to 2, from both *directions*.
  - Since $f(x)$ gets close to 4 as $\underline{x > 2 \text{ decreases}}$ to values close to 2 (i.e. 2.1, 2.01, ...,2.00001), we say that the **right-hand limit of** $f(x)$ **as** x **approaches** 2 **is** 4, which is denoted by

  $$\lim_{x \to 2^+} f(x) = 4.$$

  -Similarly, since $f(x)$ gets close to 4 as $\underline{x < 2 \text{ increases}}$ to values close to 2 (i.e. 1.9, 1.99, ...,1.99999), we say that the **left-hand limit of** $f(x)$ **as** x **approaches** 2 **is** 4, which is denoted by

  $$\lim_{x \to 2^-} f(x) = 4.$$

  We say that the **limit** of $f(x)$ is 4 as x approaches 2. An *informal* definition of **limit** is given next.

- Notice also that when $x \neq 2$, the function can be simplified as

  $$\frac{x^2 - 4}{x - 2} = \frac{(x + 2)(x - 2)}{x - 2} = x + 2,$$

  which can be plotted as a straight line with no value defined at $x = 2$.

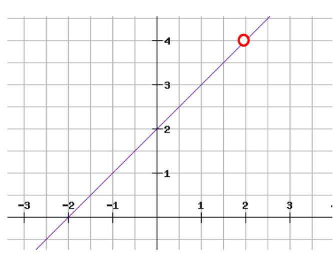


Figure 1: Plot of $f(x) = \frac{x^2-4}{x-2}$.

1.1 One-sided limits

As the name implies, with one-sided limits we will only be looking at one side of the point in question. Here are the definitions for the two one-sided limits.

Definition 1.2. (Right-handed limit) We say

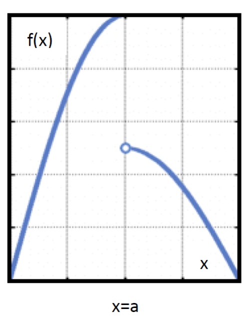
$$\lim_{x \to a^+} f(x) = L_1 \tag{2}$$

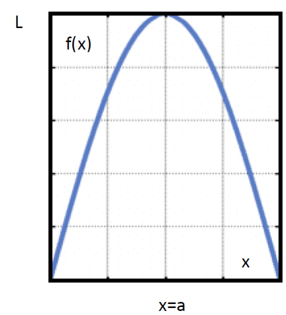
provided that we can make $f(x)$ as close to L_1 as we want for all x sufficiently close to a and $x > a$ without actually letting x be a.

Definition 1.3. (Left-handed limit) We say

$$\lim_{x \to a^-} f(x) = L_2 \tag{3}$$

provided that we can make $f(x)$ as close to L_2 as we want for all x sufficiently close to a and $x < a$ without actually letting x be a.





1.2 Two-sided limits (normal limits)

Definition 1.4. Given a function f(x), the normal limit of $f(x)$ exists and equal to L as x approaches a, i.e.

$$\lim_{x \to a} f(x) = L$$

if and only if

$$\lim_{x \to a^+} f(x) = L \qquad \text{and} \qquad \lim_{\mathrm{x} \to \mathrm{a}^-} \mathrm{f(x)} = \mathrm{L}.$$

Remarks

- For the right-handed limit we now have (note the "+") which means that we know will only look at $x > a$. Likewise for the left-handed limit we have (note the "-") which means that we will only be looking at $x < a$.

- One-sided limits don't have to exist just as normal limits are not guaranteed to exist.
- In general, the (two-sided) limit $\lim_{x\to a} f(x)$ does not exist when
  - either of one-sided limits $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^+} f(x)$ does not exist;
  - both two one-sided limits exist, but they have different values, i.e.,

$$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x).$$

- **Existence and Nonexistence:** *The existence of a limit of a function f as x approaches a (from one side of from both sides) does not depend on whether f defined at a but only on whether f is defined for x near the number a.*

Example 1.4. Consider the function

$$f(x) = x^3 - 2x + 1$$

Find $\lim_{x\to 0^-} f(x)$, $\lim_{x\to 0^+} f(x)$, and $\lim_{x\to 0} f(x)$.

Solution:

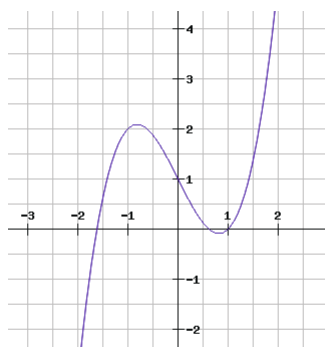


Figure 2: Plot of $f(x) = x^3 - 2x + 1$.

x $(x \to 0^-)$	$f(x) = x^3 - 2x + 1$	x $(x \to 0^+)$	$f(x) = x^3 - 2x + 1$
-0.100000000000000	1.199000000000000	0.100000000000000	0.801000000000000
-0.010000000000000	1.019999000000000	0.010000000000000	0.980001000000000
-0.001000000000000	1.001999999000000	0.001000000000000	0.998000001000000
-0.000100000000000	1.000199999999000	0.000100000000000	0.999800000001000
-0.000010000000000	1.000019999999999	0.000010000000000	0.999980000000001
-0.000000000100000	1.000000000200000	0.000000000100000	0.999999999800000
	$\vdots$		$\vdots$
Left-hand limit	$\lim_{x\to 0^-} f(x) = 1$	Right-hand limit	$\lim_{x\to 0^+} f(x) = 1$

Table 2: Values of function $f(x) = x^3 - 2x + 1$ as x approaching 0 from the left and right directions.

From the table (and the plot), That is, $\lim_{x\to 0^-} f(x) = 1$, $\lim_{x\to 0^+} f(x) = 1$, and $\lim_{x\to 0} f(x) = 1$. ■

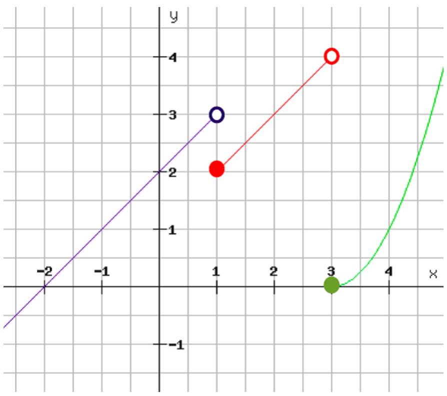
Example 1.5. Let $f(x) = \begin{cases} x+2, & \text{for} \quad x < 1 \\ x+1, & \text{for} \quad 1 \leq x < 3 \\ (x-3)^2, & \text{for} \quad x \geq 3 \end{cases}$.

Plot the function f and find the following limits, or state that it does not exist.

(a) $\lim\limits_{x\to 1^-} f(x)$, $\lim\limits_{x\to 1^+} f(x)$, $\lim\limits_{x\to 1} f(x)$

(b) $\lim\limits_{x\to 3^-} f(x)$, $\lim\limits_{x\to 3^+} f(x)$, $\lim\limits_{x\to 3} f(x)$

Solution:



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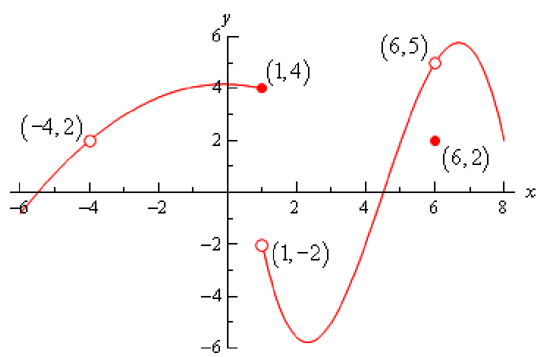
Example 1.6. Consider the following plot of a function $f(x)$. Find the following limits, or state that it does not exist.



(a) $\lim\limits_{x\to -4^-} f(x)=$ $\lim\limits_{x\to -4^+} f(x) =$

$\lim\limits_{x\to -4} f(x) =$

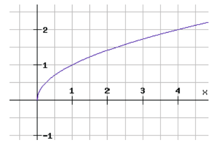
(b) $\lim\limits_{x\to 1^-} f(x)=$ $\lim\limits_{x\to 1^+} f(x) =$

$\lim\limits_{x\to 1} f(x) =$

(c) $\lim\limits_{x\to 6^-} f(x)=$ $\lim\limits_{x\to 6^+} f(x) =$

$\lim\limits_{x\to 6} f(x) =$

Example 1.7 (Right-hand limit)**.** Consider $f(x) = \sqrt{x}$. Find $\lim_{x\to 0^-} f(x)$, $\lim_{x\to 0^+} f(x)$, $\lim_{x\to 0} f(x)$ or state that it does not exist.



<u>**Solution:**</u> From the plot, we have that $\lim_{x\to 0^+} f(x) = 0$. Since the domain of x is the set of all positive number, $f(x) = \sqrt{x}$ is undefined for $x < 0$ and $\lim_{x\to 0^-} f(x)$ <u>does not exist</u>. That is, $\lim_{x\to 0} f(x)$ also <u>does not exist</u> because <u>the left-hand limit does not exist</u>. ■

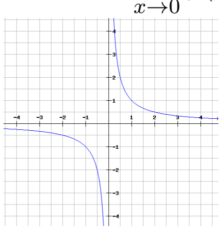
Example 1.8 (Limit that does not exist)**.** Consider $f(x) = \frac{1}{x}$. Find $\lim_{x\to 0^-} f(x)$, $\lim_{x\to 0^+} f(x)$, $\lim_{x\to 0} f(x)$ or state that it does not exist.

<u>Solution:</u>

x $(x \to 0^-)$	$f(x) = 1/x$	x $(x \to 0^+)$	$f(x) = 1/x$
-0.100000000000000	-10	0.100000000000000	10
-0.010000000000000	-100	0.010000000000000	100
-0.001000000000000	-1,000	0.001000000000000	1,000
-0.000100000000000	-10,000	0.000100000000000	10,000
-0.000010000000000	-100,000 $= -10^5$	0.000010000000000	100,000 $= 10^5$
-0.000000000100000	-10,000,000,000$= -10^{10}$	0.000000000100000	10,000,000,000$= 10^{10}$
	$\vdots$		$\vdots$
Left-hand limit does not exist	$\lim_{x\to 0^-} f(x) =$	Right-hand limit does not exist	$\lim_{x\to 0^+} f(x) =$

Table 3: Values of function $f(x) = \frac{1}{x}$ as x approaching 0 from the left and right directions.

Notice from the table that the values of $f(x)$ is unbounded and it not approaching any real number as $x \to 0^-$ nor $x \to 0^+$. That is, both left-hand and the right-hand limits do not exist as x approaches 0. Hence, $\lim_{x\to 0} f(x)$ does not exist.



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Example 1.9 (Exercise)**.** Let $f(x) = \frac{x}{|x|}$ for $x \neq 0$. Show that we can also write $f(x) = \begin{cases} -1 & , \quad x < 0 \\ 1 & , \quad x > 0 \end{cases}$.

Plot $f(x)$ and find $\lim_{x\to -1^-} f(x)$, $\lim_{x\to -1^+} f(x)$, $\lim_{x\to -1} f(x)$,

$\lim_{x\to 1^-} f(x)$, $\lim_{x\to 1^+} f(x)$, $\lim_{x\to 1} f(x)$,

$\lim_{x\to 0^-} f(x)$, $\lim_{x\to 0^+} f(x)$, $\lim_{x\to 0} f(x)$.

Example 1.10 (Important Trigonometric limits)**.** Show that

$$\lim_{x\to 0}\frac{\sin(x)}{x}=1, \qquad \lim_{x\to 0}\frac{1-\cos(x)}{x}=0.$$

Solution: This can be shown by plotting and computing the function values at x that are close to 0 as shown below.

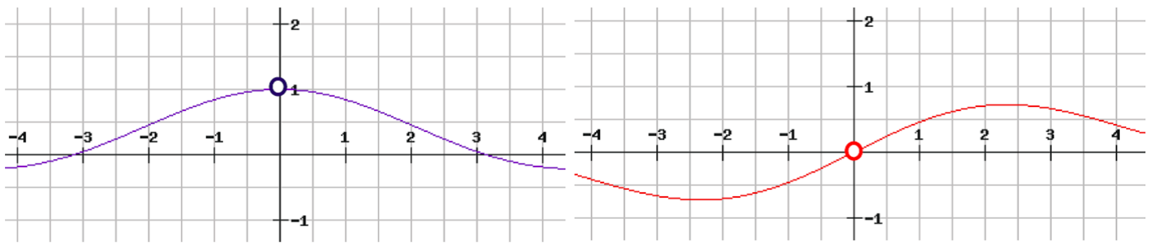

Figure 3: Left: Plot of $f(x)=\frac{\sin(x)}{x}$. Right: Plot of $f(x)=\frac{1-\cos(x)}{x}$.

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x $(x \to 0^-)$	$f(x)=\frac{\sin(x)}{x}$	x $(x\to 0^+)$	$f(x)=\frac{\sin(x)}{x}$
-0.100000000000000	0.998334166468282	0.100000000000000	0.998334166468282
-0.010000000000000	0.999983333416666	0.010000000000000	0.999983333416666
-0.001000000000000	0.999999833333342	0.001000000000000	0.999999833333342
-0.000100000000000	0.999999998333333	0.000100000000000	0.999999998333333
-0.000010000000000	0.999999999983333	0.000010000000000	0.999999999983333
	$\vdots$		$\vdots$
Left-hand limit	$\lim_{x\to 0^-} f(x)=1$	Right-hand limit	$\lim_{x\to 0^+} f(x)=1$

Table 4: Values of function $f(x)=\frac{\sin(x)}{x}$ as x approaching 0 from the left and right directions.

x $(x \to 0^-)$	$f(x)=\frac{1-\cos(x)}{x}$	x $(x\to 0^+)$	$f(x)=\frac{1-\cos(x)}{x}$
-0.100000000000000	-0.049958347219743	0.100000000000000	0.049958347219743
-0.010000000000000	-0.004999958333474	0.010000000000000	0.004999958333474
-0.001000000000000	-0.000499999958326	0.001000000000000	0.000499999958326
-0.000100000000000	-0.000049999999696	0.000100000000000	0.000049999999696
-0.000010000000000	-0.000005000000414	0.000010000000000	0.000005000000414
	$\vdots$		$\vdots$
Left-hand limit	$\lim_{x\to 0^-} f(x)=0$	Right-hand limit	$\lim_{x\to 0^+} f(x)=0$

Table 5: Values of function $f(x)=\frac{1-\cos(x)}{x}$ as x approaching 0 from the left and right directions.

2 Theorems for Computing Limit

This section will present important properties of limit, which are useful for computing limit of a function in practice.

Theorem 2.1. Assume that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Let c be any constant. Then,

1. $\lim_{x\to a} c\ f(x) = c \lim_{x\to a} f(x)$,
2. $\lim_{x\to a} [\ f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$,
3. $\lim_{x\to a} [f(x)g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right]$,
4. $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided that $\lim_{x\to a} g(x) \neq 0$,
5. $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$, where n is a positive integer.
6. $\lim_{x\to a} \left[\sqrt[n]{f(x)}\right] = \sqrt[n]{\lim_{x\to a} f(x)}$, provided that $L \geq 0$ when n is even.
7. $\lim_{x\to a} c = c$,

   In other words, the limit of a constant is just the constant. This can be seen by drawing the graph of $f(x) = c$.
8. $\lim_{x\to a} x = a$,

   As with the previous one, this can be seen by drawing the graph of $f(x) = x$.
9. $\lim_{x\to a} x^n = a^n$,

   This is really just a special case of property 5. using $f(x) = x$.

Theorem 2.2. If $\lim_{x\to a} f(x)$ exists, it is unique.

Example 2.1. Find the following limits.

1. $\lim_{x\to -8} x =$
2. $\lim_{x\to -10} 100 =$
3. $\lim_{x\to -1} (x^2 - 1) = \left(\lim_{x\to -1} x\right)^2 - \lim_{x\to -1} 1 = (-1)^2 - 1 = 0.$

■

Example 2.2. Evaluate $\lim_{x\to 1}(2x+1)^8$.
Solution:

■

Example 2.3. Evaluate $\lim_{x\to 0}\frac{x^2+3x+2}{x+1}$. [Ans: 2]
Solution: Since $\lim_{x\to 0} x^2$, $\lim_{x\to 0} x$, $\lim_{x\to 0} 2$, $\lim_{x\to 0} x$, $\lim_{x\to 0} 1$ exist and $\lim_{x\to 0} x+1 \neq 0$, then

■

Limit of a polynomial function

Let $f(x)$ be a polynomial function of degree n, i.e. $f(x)$ is in the form

$$f(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0,$$

where $c_0, c_1, \ldots c_n$ are constant. Then, for any constant real number a,

$$\lim_{x\to a^-} f(x) = f(a) \qquad \lim_{x\to a^+} f(x) = f(a) \qquad \lim_{x\to a} f(x) = f(a) \tag{4}$$

Example 2.4. Evaluate $\lim\limits_{x\to -1} \left(x^5 - x^3 + 3x^2 - 1\right)$.

Solution: ■

Example 2.5. Evaluate $\lim\limits_{x\to -1} \frac{|x^2+2x-3|}{x^2-2x+1}$.

Solution:

Since $\lim\limits_{x\to -1} (x^2 - 2x + 1) \neq 0$,

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Besides polynomial, we can use $\lim_{x\to a} f(x) = f(a)$ for other functions as shown below.

Remark: **When can we use $\lim_{x\to a} f(x) = f(a)$?**
We can use

$$\lim_{x\to a^-} f(x) = f(a) \qquad \lim_{x\to a^+} f(x) = f(a) \qquad \lim_{x\to a} f(x) = f(a) \tag{5}$$

for *any real value* a, if $f(x)$ is one of the following functions:

1. $f(x)$ is polynomial. E.g. $f(x) = x^4 + 2x^3 + x + 4$, $f(x) = x^2 - 1$.
2. $f(x) = \sin(x)$, $f(x) = \cos(x)$.
3. $f(x) = e^x$, $f(x) = c^x$ for any constant c.
4. $f(x) = \sqrt[n]{x}$ when n is an odd number.

Or, if $f(x)$ is one of the following functions with the *value of a in the domain $\mathcal{D}$*:

1. $f(x) = \sec(x)$, $f(x) = \tan(x)$ for $a \in \mathcal{D} := \{x | x \neq \frac{(2n+1)}{2}\pi, n = 0, \pm1, \pm2, \pm3, \dots\}$
2. $f(x) = \csc(x)$, $f(x) = \cot(x)$ for $a \in \mathcal{D} := \{x | x \neq 2n\pi, n = 0, \pm1, \pm2, \pm3, \dots\}$
3. $f(x) = \log_b(x)$, $f(x) = \ln(x)$ for $a \in \mathcal{D} := \{x | x > 0\}$
4. $f(x) = \sqrt[n]{x}$ when n is an even number, for $a \in \mathcal{D} := \{x | x > 0\}$.

Example 2.6. Evaluate the following limit $\lim_{x\to 0} \left(\frac{e^x - \sqrt[7]{x}}{1+\tan(x)} + \sin(x) \right)$. [Ans: 1]

Solution: Since the functions e^x, $\sqrt[7]{x}$, $\tan(x)$, and $\sin(x)$ are well-defined at $x = 0$, we can use above remark to obtain:

Theorem 2.3 (Limit that does not exist)**.** Let $\lim_{x\to a} f(x) = L \neq 0$ and $\lim_{x\to a} g(x) = 0$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

does not exist. In general, this limit is **unbounded** and will approach **infinity** in positive or negative direction ,i.e. $\lim_{x\to a^-} \frac{f(x)}{g(x)} = \pm\infty$, or $\lim_{x\to a^+} \frac{f(x)}{g(x)} = \pm\infty$.

Example 2.7. Evaluate (a) $\lim_{x\to 3} \frac{x}{3-x}$. (b) $\lim_{x\to 3^-} \frac{3-x}{x}$

Solution:

■

Remark: $\boxed{\text{Computing } \lim_{x\to a} \frac{f(x)}{g(x)}}$

Let f and g be functions and suppose $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist with

$$\lim_{x\to a} f(x) = L_1, \qquad \lim_{x\to a} g(x) = L_2.$$

We consider 3 cases of $\lim_{x\to a} \frac{f(x)}{g(x)}$:

(I) If $L_2 \neq 0$, $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}$.

(II) If $L_1 \neq 0$ and $L_2 = 0$, $\lim_{x\to a} \frac{f(x)}{g(x)}$ **does not exist**.
In this case, $\lim_{x\to a} \frac{f(x)}{g(x)}$ is either ∞ or $-\infty$. $\Longrightarrow$ *"Infinity Limits "*

(III) If $L_1 = 0$ and $L_2 = 0$, $\Longrightarrow$ $\boxed{\text{" Indeterminate Form } \frac{0}{0} \text{"}}$
To find $\lim_{x\to a} \frac{f(x)}{g(x)}$, we have to rearrange $\frac{f(x)}{g(x)}$, e.g.

- factor $f(x)$, $g(x)$ and eliminate the common terms before taking limit
- use *rationalization*: multiply the same term to both $f(x)$, $g(x)$ to use

$$(p-q)(p+q) = p^2 - q^2.$$

Note: There are many more kinds of indeterminate forms we will be discussing later.

Useful formulas: $\boxed{p^3 + q^3 = (p+q)(p^2 - pq + q^2)}$; $\boxed{p^3 - q^3 = (p-q)(p^2 + pq + q^2)}$;
and $\boxed{(p+q)^3 = p^3 + 2p^2q + 2pq^2 + q^3}$; $\boxed{(p-q)^3 = p^3 - 2p^2q + 2pq^2 - q^3}$.

Example 2.8. Evaluate

$$(a) \lim_{x \to -1} \frac{x^3 + 2x^2 + x}{x^2 - 4x - 5}. \qquad (b) \lim_{x \to -3^+} \frac{x-3}{x^2 - 6x + 9}.$$

[Ans: 0, DNE] Note(a): $x^2 - 4x - 5 = (x+1)(x-5)$
Solution:

$\blacksquare$

Example 2.9. Evaluate
$$\lim_{x\to 0} \frac{\sqrt{x^2+4}-2}{x^2}$$
[Ans: 1/4]
<u>Solution:</u>

$\blacksquare$

Example 2.10. Evaluate $\lim\limits_{x\to -1} \frac{1-x^{2/3}}{1+x}$.
<u>Solution:</u>

$\blacksquare$

Example 2.11. Let $f(x) = \begin{cases} \frac{x^2+3x+2}{|x+1|} & , \quad x < -1 \\ 2 - x^2 & , \quad x \geq -1 \end{cases}$. Evaluate

$$(i) \lim_{x \to -1^-} f(x), \qquad (ii) \lim_{x \to -1^+} f(x), \qquad (iii) \lim_{x \to -1} f(x).$$

Solution:

$\blacksquare$

2.1 Exercise

1. Evaluate (a) $\lim\limits_{x\to 0} \frac{3}{x}$, (b) $\lim\limits_{x\to 0} \frac{2(x-3)^2-18}{x}$ (c) $\lim_{x\to -1} \left(\frac{x^2+2}{x+1} - \frac{3x}{-1-x} \right)$.

2. Let $f(x) = \begin{cases} \frac{x^2+3x+2}{x+1} & , \quad x \neq -1 \\ 3 & , \quad x = -1 \end{cases}$. Plot $f(x)$ and find

   (i) $\lim\limits_{x\to -1^-} f(x)$, (ii) $\lim\limits_{x\to -1^+} f(x)$, (iii) $\lim\limits_{x\to -1} f(x)$.

3. Let $f(x) = \begin{cases} \frac{x^2-1}{x-1} & , \quad x < 1 \\ x^2 & , \quad x \geq 1 \end{cases}$. Plot $f(x)$ and find

   (i) $\lim\limits_{x\to 1^-} f(x)$, (ii) $\lim\limits_{x\to 1^+} f(x)$, (iii) $\lim\limits_{x\to 1} f(x)$.

4. Find $\lim\limits_{x\to 0^-} \frac{\sqrt{x^2+4}-2}{x^2}$.

5. (*) Find $\lim\limits_{x\to -1^+} \frac{x^3+x^2}{|x^2-4x-5|}$ and $\lim\limits_{x\to -1} \frac{x^3+x^2}{|x^2-4x-5|}$.

Compute the following limits.

1. (a) $\lim\limits_{x\to 25} \frac{5-\sqrt{x}}{25-x}$ (b) $\lim\limits_{x\to 2} \frac{|x-2|}{x-2}$

2. (a) $\lim\limits_{x\to 0} \frac{x^2}{\sqrt{x^2+12}-\sqrt{12}}$ (b) $\lim\limits_{x\to -2} \frac{|x|-2}{\sqrt{x^2-2x}-\sqrt{8}}$

3. $\lim\limits_{x\to 0} \frac{x\sin(x/2)}{|x|}$

3 Trigonometric Limits

This section considers the limits of trigonometric functions, such as $\sin(x), \cos(x), \tan(x),$.

Example 3.1. Show that the limit $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist.
Solution: Notice that

$$\sin\left(\frac{1}{x}\right) = \begin{cases} -1, & \text{for } 1/x = \frac{\pi}{2} + n\pi, \text{ if } n \text{ is odd} \\ 1, & \text{for } 1/x = \frac{\pi}{2} + n\pi, \text{ if } n \text{ is even} \end{cases}$$

and therefore f oscillates between -1 and 1 as $x \to 0$. E.g. if n is odd (e.g. $n = 501$ or $x \approx 0.00063$), $\sin\left(\frac{1}{x}\right) = -1$ and if n is even (e.g. $n = 500$ or $x \approx 0.00064$), $\sin\left(\frac{1}{x}\right) = 1$. Note that when n is large, the magnitude of x is small and close to zero. That is, it is impossible for $\sin\left(\frac{1}{x}\right)$ to converge to any real number as x approaches 0. ■

Although the limit $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist, $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$ exist and can be found by the **Squeeze Theorem**. This theorem is also known by other names such as **Sandwiching theorem** and **Pinching theorem**.

3.1 Squeeze Theorem

Theorem 3.1. Let f and g be functions with $f(x) \leq g(x)$ for all x on an open interval that contains a number c (except possibly at $x = c$ itself) then,

$$\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x). \tag{6}$$

Theorem 3.2 (**Squeeze Theorem**)**.** Let f, g, and h be functions with

(i) $f(x) \leq h(x) \leq g(x)$ for all x on an open interval that contains a number c (except possibly at $x = c$ itself) and

(ii) $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = L$.

Then,

$$\lim_{x\to c} h(x) = L. \tag{7}$$

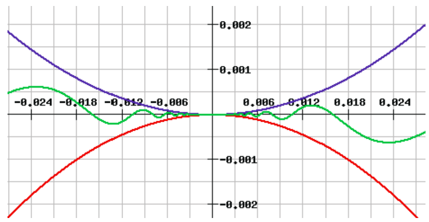


Figure 4: Illustration for Squeeze Theorem

Example 3.2. Suppose $-x^2 - 2x + 2 \leq g(x) \leq (x+1)^2 + 3$ for all real numbers x.
Determine $\lim\limits_{x \to -1} g(x)$.

Solution:

Example 3.3. Find $\lim\limits_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$ if it exists.

Solution: Note that $\lim\limits_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) \neq (\lim\limits_{x \to 0} x^2)(\lim\limits_{x \to 0} \sin\left(\frac{1}{x}\right))$ since $\lim\limits_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist.
The Squeeze theorem can be used as follows. For $x \neq 0$, since $\lim\limits_{x \to 0} x^2 = 0$ and $\lim\limits_{x \to 0} -x^2 = 0$ with

$$\begin{aligned} -1 \leq &\quad \sin(\tfrac{1}{x}) &\leq 1 \\ -x^2 \leq &\quad x^2 \sin(\tfrac{1}{x}) &\leq x^2, \end{aligned}$$

then $\lim\limits_{x \to 0} x^2 \sin(\frac{1}{x}) = 0$. This example is also illustrated through the figure below.

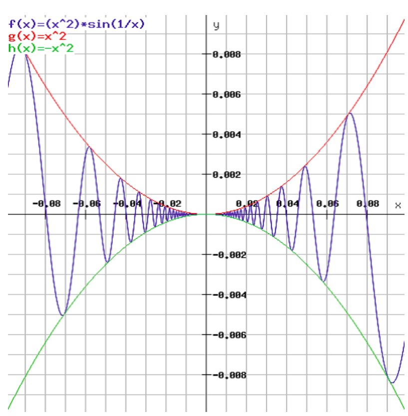


Figure 5: Plots of functions $-x^2$, $x^2 \sin\left(\frac{1}{x}\right)$, and x^2. ■

Example 3.4. (Exercise) Find $\lim\limits_{x \to 0} x \sin\left(\frac{1}{x}\right)$ if it exists.

Example 3.5. Find $\lim_{x \to 0} |x| \cos\left(\frac{1}{2x}\right)$ if it exists.

Solution:

$\blacksquare$

3.2 Finding Trigonometry Limits by using Substitution and Identities

Recall the following important trigonometric limits

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

We can use this to find other trigonometric limits by using substitution together with some identities.

Example 3.6. Find the limit

$$\lim_{x \to 0} \frac{\sin(2x)}{5x}.$$

Solution:

Example 3.7. Find the limit

$$\lim_{x \to 0} \frac{\tan(3x) + x^2 - 4x}{x}.$$

Solution:

Example 3.8. Find the limit

$$\lim_{x\to 2} \frac{\sin(x-2)}{x^2-3x+2}.$$

Solution:

Example 3.9. (Exercise) Find the limit

$$\lim_{x\to 3} \frac{\sin(x-2)}{x-2}.$$

Solution:

Example 3.10. (Exercise) Evaluate

$$(a)\ \lim_{x\to 0} \frac{1-\cos(x)}{x}, \qquad (b)\ \lim_{x\to 0} \frac{1-\cos(x)}{x\sin(x)}.$$

Solution: Use *rationalization.*

4 Limits that involve infinity

We will consider the following 2 cases of limits that involve infinity.

- Infinity Limits: $\lim_{x\to a} f(x) = -\infty$ or $\lim_{x\to a} f(x) = \infty$.
- Limits at infinity: $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$.

4.1 Infinity Limits: $\lim_{x\to a} f(x) = -\infty$ or $\lim_{x\to a} f(x) = \infty$

Definition 4.1. We say

$$\lim_{x\to a} f(x) = \infty \tag{8}$$

if we can make $f(x)$ arbitrarily large for all x sufficiently close to a, from both sides, *without* actually letting $x = a$.
We say

$$\lim_{x\to a} f(x) = -\infty \tag{9}$$

if we can make $f(x)$ arbitrarily large and negative for all x sufficiently close to a, from both sides, without actually letting $x = a$.

The following figure illustrate the definition of infinity limits given above.

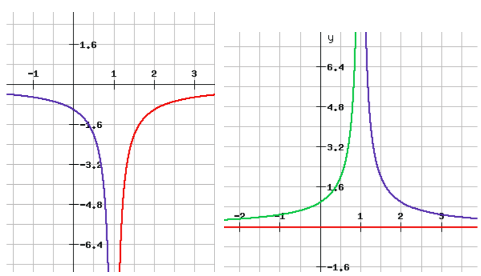


Figure 6: Left: $\lim_{x\to a} f(x) = -\infty$; Right: $\lim_{x\to a} f(x) = \infty$ where $a = 1$.

Remark: These definitions can be appropriately modified for the one-sided limits as well. In particular, any of the following limits is called an **infinite limit** as shown in Figures 6 and 7. Note that **none of these limits exists**.

$$\begin{array}{lll} \lim_{x\to a^-} f(x) = -\infty & \lim_{x\to a^+} f(x) = -\infty & \lim_{x\to a} f(x) = -\infty \\ \lim_{x\to a^-} f(x) = \infty & \lim_{x\to a^+} f(x) = \infty & \lim_{x\to a} f(x) = \infty. \end{array}$$

We can next define a vertical asymptote in terms of infinity limit as follows.

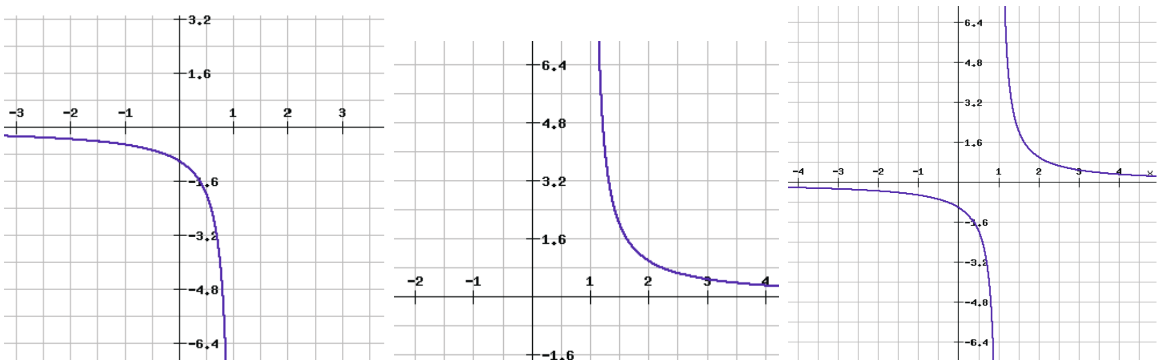


Figure 7: The illustration of the functions with (i) $\lim\limits_{x\to a^-} f(x) = -\infty$, (ii) $\lim\limits_{x\to a^+} f(x) = \infty$ and (iii) $\lim\limits_{x\to a^-} f(x) = -\infty$ $\lim\limits_{x\to a^+} f(x) = \infty$, for $a = 1$.

Definition 4.2. The function $f(x)$ will have a <u>*vertical asymptote*</u> at $x = a$ if <u>any</u> of the following limits at $x = a$ is true.

$$\begin{array}{lll} \lim\limits_{x\to a^-} f(x) = -\infty & \lim\limits_{x\to a^+} f(x) = -\infty & \lim\limits_{x\to a} f(x) = -\infty \\ \lim\limits_{x\to a^-} f(x) = \infty & \lim\limits_{x\to a^+} f(x) = \infty & \lim\limits_{x\to a} f(x) = \infty. \end{array} \tag{10}$$

Example 4.1. Let a be a constant real number and n be a positive integer.

(a)

$$\lim_{x\to a} \frac{1}{(x-a)^n} = \infty, \qquad \text{where } n \text{ is even.}$$

(b)

$$\lim_{x\to a^+} \frac{1}{(x-a)^n} = \infty, \qquad \lim_{x\to a^-} \frac{1}{(x-a)^n} = -\infty, \qquad \text{where } n \text{ is odd.}$$

Note that both (a) and (b) implies that $x = a$ is the vertical asymptote of $\frac{1}{(x-a)^n}$. ■

Consider a general rational function, which is in the form

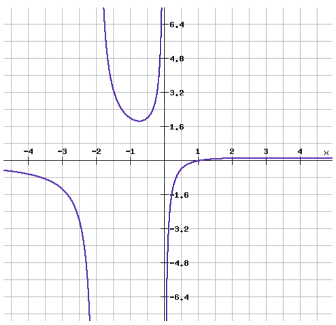
$$f(x) = \frac{p(x)}{q(x)}$$

where p and q have no common factors. Then, if q contains a factor of the form $(x-a)^n$ where n is a positive integer, then $x = a$ is a vertical asymptote, since at least one of the following limits in (10) is true. Note that, in this case, $p(a) \neq 0$.

Example 4.2. Determine the vertical asymptotes of the rational function

$$f(x) = \frac{x-1}{x^2+2x}.$$

Solution



Theorem 4.1. Given functions $f(x)$ and $g(x)$ with, $\boxed{\lim_{x\to a} f(x) = \infty}$ and $\boxed{\lim_{x\to a} g(x) = L}$

for some (finite) real numbers a and L. Then,

1. $\lim_{x\to a} [f(x) \pm g(x)] = \infty$.
2. For $L > 0$, $\lim_{x\to a} [f(x)g(x)] = \infty$. For $L < 0$, $\lim_{x\to a} [f(x)g(x)] = -\infty$.
3. $\lim_{x\to a} \frac{g(x)}{f(x)} = 0$.

Note: The above set of facts also holds for one-sided limits. They will also hold if , with a change of sign on the infinities in the first three parts.

4.1.1 Exercise

1. Find (i) $\lim_{x\to 0^-} \frac{1}{|x|}$, (ii) $\lim_{x\to 0^+} \frac{1}{|x|}$, (iii) $\lim_{x\to 0} \frac{1}{|x|}$.
2. Find (i) $\lim_{x\to -2^-} \frac{-4}{x+2}$, (ii) $\lim_{x\to -2^+} \frac{-4}{x+2}$, (iii) $\lim_{x\to -2} \frac{-4}{x+2}$.
3. Find (i) $\lim_{x\to -2^-} \frac{-4x}{(x+2)^2}$, (ii) $\lim_{x\to -2^+} \frac{-4x}{(x+2)^2}$, (iii) $\lim_{x\to -2} \frac{-4x}{(x+2)^2}$.
4. Find (i) $\lim_{x\to \frac{\pi}{2}^+} \tan(x)$, (ii) $\lim_{x\to \frac{\pi}{2}^-} \tan(x)$, (iii) $\lim_{x\to \frac{\pi}{2}} \tan(x)$.
5. Find $\lim_{x\to 0^+} \ln(x)$.

4.2 Limits at Infinity: $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$

Definition 4.3. If a function f approaches a constant value L_1 as the independent variable x increases without bound, i.e. $x \to \infty$, then we write

$$\lim_{x\to\infty} f(x) = L_1.$$

Similarly, if a function f approaches a constant value L_2 as the independent variable x decreases without bound, i.e. $x \to -\infty$, then we write

$$\lim_{x\to-\infty} f(x) = L_2.$$

In any of these cases, we say that f has a **limit at infinity**.

Remarks: The followings are the possibilities for limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

(i) One limit exists but the other does not.

(ii) Both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ exist, and equal the same number.

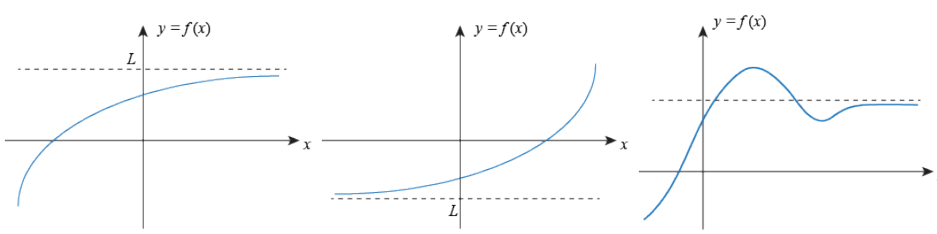
(iii) Both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ exist, but they have different values.

(iv) Neither $\lim_{x\to\infty} f(x)$ nor $\lim_{x\to-\infty} f(x)$ exists.

If any of these cases (i)-(iii) happens, then the graph of f has a **horizontal asymptote**, which will be defined formally next.

Definition 4.4. A line $y = L$ is said to be a **horizontal asymptote** for the graph of a function f if

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L.$$



Theorem 4.2. [Limits at Infinity] If r is a positive rational number and a is any real number then,

$$\lim_{x\to\infty} \frac{1}{(x-a)^r} = 0 \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{1}{(x-a)^r} = 0.$$

Example 4.3. From above theorem, we have $\lim\limits_{x\to\infty} \frac{1}{x-1} = 0$ and $\lim\limits_{x\to-\infty} \frac{1}{x-1} = 0$, which implies that $x = 0$ is a horizontal asymptote as shown in the figure below.

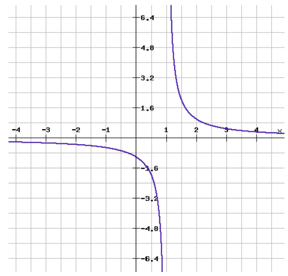


Figure 8: Plot of $f(x) = \frac{1}{x-1}$.

Example 4.4. Determine all vertical and horizontal lines of the function

$$f(x) = \frac{3+\sqrt{x}}{\sqrt{x}-5}.$$

Explain your answer by using limits of $f(x)$. ■

Solution:

Notice that the domain of $f(x)$ is ....

4.2.1 Limits at infinity of rational functions

We will consider the limit of a function that can be written as a fraction of 2 functions $p(x)$ and $q(x)$. The following is what we have so far.

Remark Let L be any constant real number. For $b = a, -\infty, \infty$ (a is a constant real number),

1. if $\left\{ \begin{array}{l} \lim_{x\to b} p(x) = L \\ \lim_{x\to b} q(x) = \pm\infty \end{array} \right\} \Longrightarrow \lim_{x\to b} \frac{p(x)}{q(x)} = 0$

2. if $\left\{ \begin{array}{l} \lim_{x\to b} p(x) = \infty \\ \lim_{x\to b} q(x) = L \neq 0 \end{array} \right\} \Longrightarrow \lim_{x\to b} \frac{p(x)}{q(x)} = \left\{ \begin{array}{ll} -\infty, & L < 0 \\ \infty, & L > 0 \end{array} \right.$

   if $\left\{ \begin{array}{l} \lim_{x\to b} p(x) = -\infty \\ \lim_{x\to b} q(x) = L \neq 0 \end{array} \right\} \Longrightarrow \lim_{x\to b} \frac{p(x)}{q(x)} = \left\{ \begin{array}{ll} \infty, & L < 0 \\ -\infty, & L > 0 \end{array} \right.$

3. if $\left\{ \begin{array}{l} \lim_{x\to b} p(x) = \pm\infty \\ \lim_{x\to b} q(x) = \pm\infty \end{array} \right\} \Longrightarrow \lim_{x\to b} \frac{p(x)}{q(x)}$ has Intermediate form: $\frac{\pm\infty}{\pm\infty} \Rightarrow$ rearrange in a new form.

The last case often occur when $\frac{p(x)}{q(x)}$ is a rational function. Recall that **rational** functions are in the form

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x) \neq 0$ are **polynomials**. We can compute limit at infinity by looking at the **end behavior** of $f(x)$ as $|x|$ gets large, which will reflect in the **highest degree** of each polynomial.

Theorem 4.3. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ be a polynomial of degree $n \geq 0$ (i.e. $a_n \neq 0$), and let $q(x) = b_n x^m + b_{m-1}x^{m-1} + \cdots + b_2x^2 + b_1x + b_0$ be a polynomial of degree $m \geq 0$ (i.e. $b_m \neq 0$). Then, for the **rational** function $f(x) = \frac{p(x)}{q(x)}$

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \lim_{x\to\infty} \left(\frac{a_n x^n}{b_m x^m} \right) = \left(\frac{a_n}{b_m} \right) \lim_{x\to\infty} x^{n-m}$$

$$\lim_{x\to-\infty} \frac{p(x)}{q(x)} = \lim_{x\to-\infty} \left(\frac{a_n x^n}{b_m x^m} \right) = \left(\frac{a_n}{b_m} \right) \lim_{x\to-\infty} x^{n-m}.$$

That is, when we want to take a limit at infinity for a polynomial then all we need to really do is look at the term with the largest power and ask what that term is doing in the limit since the polynomial will have the same behavior.

Example 4.5. Evaluate

$$\lim_{x\to\infty} \frac{1+2x^2-3x^3}{2x^3+3x}.$$

Solution:

Alternative approach:

Example 4.6. Evaluate

$$\lim_{x\to-\infty} \sqrt{\frac{x^5-3x^3-7}{4x^5-x}}.$$

Solution:

Example 4.7. Evaluate

$$\lim_{x\to-\infty} \frac{x^5-x^2-4}{x^7-2}.$$

Solution:

Example 4.8. Evaluate

$$\lim_{x\to-\infty} \frac{x^7-2}{x^5-x^2-4}.$$

Solution:

Example 4.9 (Exercise)**.** Determine horizontal asymptotes (if any) of the following functions.

(a) $f(x) = \frac{2}{e^{-x}+1}$

(b) $f(x) = \frac{x}{\sqrt{2+x^2}}$

Remark 2: Indeterminate form "$\infty - \infty$" $\Longrightarrow$ Use *"Rationalization"*.

Example 4.10. Find the limit $\lim_{x\to\infty}(\sqrt{x^2+5x} - x)$

$$\begin{aligned}
\lim_{x\to\infty}(\sqrt{x^2+5x} - x) &= \lim_{x\to\infty}(\sqrt{x^2+5x} - x) \cdot \left(\frac{\sqrt{x^2+5x}+x}{\sqrt{x^2+5x}+x}\right) \\
&= \lim_{x\to\infty}\left(\frac{x^2+5x-x^2}{\sqrt{x^2+5x}+x}\right) \\
&= \lim_{x\to\infty}\left(\frac{5x}{\sqrt{x^2+5x}+x}\right) \\
&= \lim_{x\to\infty}\left(\frac{5x}{\sqrt{x^2(1+\frac{5}{x})}+x}\right) = \lim_{x\to\infty}\left(\frac{5x}{|x|\sqrt{(1+\frac{5}{x})}+x}\right) \\
&= \lim_{x\to\infty}\left(\frac{\frac{5x}{|x|}}{\sqrt{(1+\frac{5}{x})}+\frac{x}{|x|}}\right) \\
&= \frac{5}{\sqrt{1+0}+1} = \frac{5}{2}
\end{aligned}$$

where we have used the facts that $\sqrt{x^2} = |x|$ and$x \to \infty \Longrightarrow x > 0 \Rightarrow |x| = x$.

Example 4.11. Evaluate

$$\lim_{x\to\infty}(x^2 - \sqrt{x^4+7x^2+1}).$$

[Ans: -7/2]
Solution:

4.2.2 Exercise

1. Find $\lim_{x\to\infty} 2x^4 - x^2 - 8x$.

2. Find $\lim_{x\to-\infty} \frac{1}{4}x^7 - x^5 + x^2 + 3$.

3. Find $\lim_{x\to\infty} \frac{2x^4-x^2-8x}{1-3x^4}$ and $\lim_{x\to-\infty} \frac{2x^4-x^2-8x}{1-3x^4}$.

4. Find $\lim_{x\to\infty} \frac{\sqrt{x^2-8}}{1+2x}$ and $\lim_{x\to-\infty} \frac{\sqrt{x^2-8}}{1+2x}$ (note: $\sqrt{x^2} = |x|$).

5. Find $\lim_{x\to-\infty} \frac{x^2-8}{1-2x}$ and $\lim_{x\to\infty} \frac{1-2x}{x^2-8}$.

4.2.3 Exercise (for non-rational functions)

1. Find $\lim_{x\to\infty} e^x$, $\lim_{x\to\infty} e^{-x}$, $\lim_{x\to-\infty} e^x$, $\lim_{x\to-\infty} e^{-x}$.

2. Find $\lim_{x\to\infty} e^{1-3x-5x^2}$.

3. Find $\lim_{x\to\infty} e^{1/x}$ and $\lim_{x\to 0^+} e^{1/x}$.

4. Find $\lim_{x\to\infty} \frac{3e^{5x}+2e^{-2x}}{e^{-2x}-e^{5x}}$ and $\lim_{x\to-\infty} \frac{3e^{5x}+2e^{-2x}}{e^{-2x}-e^{5x}}$

5. Find $\lim_{x\to-\infty} \ln\left(\frac{1}{t^2-t}\right)$.

6. Find $\lim_{x\to\infty} \arctan(x)$ and $\lim_{x\to 0^-} \arctan\left(\frac{1}{x}\right)$.

5 Continuous Functions

Definition 5.1. (Continuity)

- A function $f(x)$ is said to be **continuous** at $x = a$ if
  (1) $f(a)$ is well-defined,
  (2) $\lim\limits_{x \to a} f(x)$ exists, and
  (3) $\lim\limits_{x \to a} f(x) = f(a)$.
- A function $f(x)$ is said to be **continuous** on an open interval (a, b) if it is continuous at each point in the interval.
- A function $f(x)$ is said to be **continuous** on a closed interval $[a, b]$ if it is continuous at each point in the interval (a, b) and

$$\lim_{x \to a^+} f(x) = f(a) \qquad \lim_{x \to b^-} f(x) = f(b).$$

Example 5.1. Let

$$f(x) = \begin{cases} \frac{x^2+3x+2}{x+1} & , \ x < -1 \\ 3 & , \ x = -1 \\ x^2 & , \ -1 < x \leq 2 \\ x & , \ x > 2 \end{cases}$$

Determine whether the piecewise function above is continuous at $x = -1$ and $x = 2$.
Solution: To determine the continuity at $x = -1$, we will consider $f(-1)$ and the following limits.

$$(i) \lim_{x \to -1^-} f(x), \qquad (ii) \lim_{x \to -1^+} f(x), \qquad (iii) \lim_{x \to -1} f(x).$$

<u>Remark</u> If $f(x)$ is continuous at a , then

$$\lim_{x\to a^-} f(x) = f(a) \qquad \lim_{x\to a^+} f(x) = f(a) \qquad \lim_{x\to a} f(x) = f(a).$$

Example 5.2. Let

$$f(x) = \begin{cases} \frac{(x-2)^2}{x^2-4} + 2k & , \quad x > 2 \\ h & , \quad x = 2 \\ 2x + k & , \quad x < 2 \end{cases}$$

where h and k are constant real numbers. Find h and k that make f continuous at $x = 2$.

<u>Solution:</u>

Example 5.3. (Exercise) For each of the following functions, determine the largest interval on which the function is continuous.

$$(i)\ f(x) = \sqrt{x-1} \qquad (ii)\ f(x) = \sqrt{1-x^2} \qquad (iii)\ f(x) = \frac{1}{\sqrt{1-x^2}}$$

[ANS: $[1,\infty]$, $[-1,1]$, $(-1,1)$]

Theorem 5.1. If the function f and g are continuous at a number a, then
- the sum $f+g$,
- the product fg and
- the quotient f/g, for $g(a) \neq 0$,

are continuous at $x = a$.

Definition 5.2. Let $f(x)$ be a function.

(1) $f(x)$ has an *infinite discontinuity* at a if $x = a$ is a vertical asymptote for the graph $y = f(x)$.

(2) $f(x)$ has a *finite discontinuity* (or *jump discontinuity*) at a if

$$\lim_{x\to a^-} f(x) = L_1 \quad \text{and} \quad \lim_{x\to a^+} f(x) = L_2 \quad \text{but} \quad L_1 \neq L_2.$$

(3) $f(x)$ has *removable discontinuity* at a if $\lim_{x\to a} f(x)$ exists but, either

- f is not defined at $x = a$, or
- $f(a)$ is well-defined, but $f(a) \neq \lim_{x\to a} f(x)$.

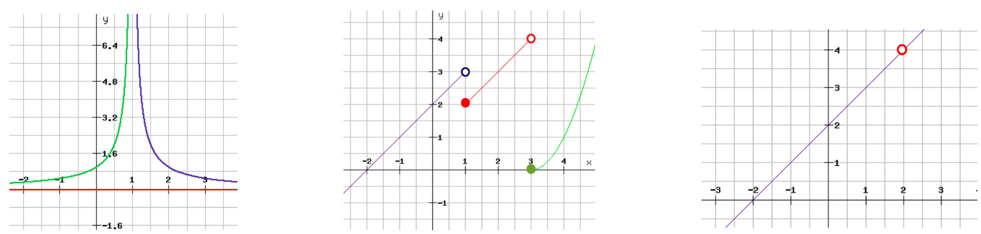


Figure 9: Examples of (1) infinite discontinuity; (2)jump discontinuity; (3)removable discontinuity.

Theorem 5.2 (Composite Functions)**.** :

- (Limit)

  If $\lim_{x\to a} g(x) = b$ and $f(x)$ is continuous at b then, the *composite function* $(f \circ g)(x) = f(g(x))$ is continuous at a and

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)).$$

- (Continuity) If g is continuous at a number a and f is continuous at $g(a)$, then the composite function $(f \circ g)(x) = f(g(x))$ is continuous at a.

Example 5.4. Determine the limit

$$\lim_{x\to\infty} \sin\left(\frac{1}{x}\right).$$

Solution: Let $f(x) = \sin(x)$ and $g(x) = \frac{1}{x}$. Since

$$\lim_{x\to\infty} \frac{1}{x} = \text{.................}$$

and $\sin(x)$ is continuous at every real number x. Then

$$\lim_{x\to\infty} \sin\left(\frac{1}{x}\right) = \lim_{x\to\infty} f(g(x)) = f(\lim_{x\to\infty} g(x)) = \text{................................}$$

Example 5.5. (Exercise) Let $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$. Determine the largest interval on which the composite function $(f \circ g)(x)$ is continuous.
[ANS: $(-\infty, \infty)$]

Solution:

Theorem 5.3. (**Intermediate Value Theorem**)
Suppose that $f(x)$ is continuous on $[a, b]$ and let M be any number between $f(a)$ and $f(b)$. Then there exists a number $c \in [a, b]$ such that,

$$f(c) = M.$$

Example 5.6. Show that $p(x) = 2x^3 - 5x^2 - 10x + 5$ has a root somewhere in the interval $[-1, 2]$.
Solution: [Note that $p(-1) = 8, p(2) = -19$]

5.1 Exercises

1. Find $\lim_{x\to 0} e^{-\cos(x)}$.
2. Determine where the function $f(x) = \frac{x^2+2x+1}{x^2-1}$ is not continuous.
3. Use the intermediate value theorem to show that $p(x) = x^2 - x - 5$ has a root somewhere in the interval $[-1, 4]$.

6 Formal Definition of Limit and Continuity

Definition 6.1. **ε-δ ("epsilon-delta") definition of limit:**
Let $f(x)$ be a function defined on an interval that contains $x = a$, except possibly at $x = a$. Then

$$\lim_{x\to a} f(x) = L$$

if for every number ε there is some number δ such that

$$|f(x) - L| < \varepsilon \qquad \text{whenever} \qquad |\mathrm{x} - \mathrm{a}| < \delta.$$

Definition 6.2. **ϵ-δ ("epsilon-delta") definition of continuity:**
Let $f(x)$ be a function defined on an interval that contains $x = a$. Then f(x) is *continuous* at a if for every number ε there is some number δ such that

$$|f(x) - f(a)| < \varepsilon \qquad \text{whenever} \qquad |\mathrm{x} - \mathrm{a}| < \delta.$$