Inequalities

Let a be a real number. Then either

$$a \text{ is negative: } \quad a < 0, \quad a \text{ is zeros: } \quad a = 0, \text{ or } \quad a \text{ is positive: } \; a > 0.$$

Let a and b be real numbers. Then either

$$a < b \quad (a - b \text{ is negative}), \quad a = b \quad (a - b \text{ is zero}), \text{ or } \quad a > b \;\; (a - b \text{ is is positive}).$$

1 Properties

(1) If $a < b$, then $a + c < b + c$ and $a - c < b - c$.

(2) If $a < b$ and $c > 0$ then $ac < bc$.
But if $a < b$ and $c < 0$ then $ac > bc$.
Note that mulitplication by a negative number flips the inequality.

(3) If $a < b$ and $b < c$, then $a < c$.

(4) If $a > 0$, then $\frac{1}{a} > 0$.

(5) If $0 < a < b$, then $\frac{1}{b} < \frac{1}{a}$

(6) If $0 < a < b$, then $\frac{1}{b^n} < \frac{1}{a^n}$ for any positive integer n.

(7) If $0 \leq a < b$, then $a^n < b^n$ for any positive integer n.

(8) If $0 < a < b$ and $0 < c < d$, then $0 < ac < bd$.

Example Find all real numbers x that satisfy the following inequalities.

1. $x + 5 \leq x - 2$
2. $5 - 2x > -5 - 2x$
3. $5x - 2 < 5x - 2$
4. $\frac{2x}{3} + 5 \leq x - 1$

Example Let a, b be two real numbers such that $a + 3 \leq b + 1 < 0$. Show that

$$2ab \leq ab^2 - a^2b.$$

Example

Find all real numbers x such that

$$\frac{2}{x-1} \geq 5$$

Inequality with quadratic polynomial

When the inequality involves quadratic polynomial $ax^2 + bx + c$, if $b^2 - 4ac > 0$ it is useful to use the following factorization:

$$ax^2 + bx + c = a(x - K_1)(x - K_2), \quad K_1 = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) \; K_2 = \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

If $b^2 - 4ac < 0$, we cannot factor of the polynomial in terms of real numbers. Moreover,

- if $a > 0$, then we always have $ax^2 + bx + c > 0$,
- if $a < 0$, then we always have $ax^2 + bx + c < 0$.

Example Solve the following inequities.

1. $x^2 + 8 > 6x$
2. $3x^2 - x + 5 < 0$
3. $-x^2 + 2x \leq 4$
4. $x^3 + x^2 \geq 2x$
5. $\left(-2x^2 - x - 4\right)\left(x^2 + 2x + 1\right)$

Example(Continued)