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EE431 Economics of Financial Markets and Institutions
Problem Set 5

Solution

1. $ER_A = 0.05 + 0.5F_1$; asset A has $\beta_{A1} = 0.5$.

   $ER_B = 0.02 - 0.25F_1$; asset B has $\beta_{B1} = -0.25$.

   Notation: β_{j1} and β_{j1} for j = A; B denote the responses of the rates of return on assets A and B to the factor 1 (F_1).

   (a) Determine the portfolio weights you need to place on A and B in order to construct a portfolio which has zero exposure to the factor 1 (F_1) (a risk-free portfolio). What is the expected return of the risk-free portfolio?

   Solution.

   Let w_A be the weight of asset A and w_B be the weight of asset B in a portfolio. Then, expected return of the portfolio can be written as follows.

$$
\begin{aligned}
E(R_p) &= aE(X) + bE(Y), \\
E(R_p) &= w_A E(R_A) + w_B E(R_B), \\
&= w_A(0.05 + 0.5F_1) + w_B(0.02 - 0.25F_1), \\
&= (0.05w_A + 0.02w_B) + (0.5w_A - 0.25w_B)F_1.
\end{aligned}
$$

   A riskless portfolio must have beta equal to zero. Therefore,

   $0.5w_A - 0.25w_B = 0 \Rightarrow$ equation (1).

   The sum of the weights must always be 1.

   $w_A + w_B = 1 \Rightarrow$ equation (2),

   Solving the system of equations (equation (1) and equation (2)) for w_A and w_B and get $w_A = \frac{1}{3}$ and $w_B = \frac{2}{3}$.

   Substitute $w_A = \frac{1}{3}$ and $w_B = \frac{2}{3}$ in to $E(R_p) = (0.05w_A + 0.02w_B) + (0.5w_A - 0.25w_B)F_1$ and get $E(R_p) = \frac{0.05 + 2(0.04)}{3} = 0.03$.

   Hence, to construct a riskless portfolio, the weight placed on asset A must be equal to $\frac{1}{3}$ and the weight placed on asset B must be equal to $\frac{2}{3}$. The return of the risk-free portfolio is then equal to 0.03 or 3%.

   (a) From (a), is there any abritrage opportunity exist if the market risk-free interest rate is 1%? If yes, explain how an investor can make an arbitrage profit.

   Solution.

   The risk-free portfolio in (a) and the risk-free asset avaiable in the market (which pays the market risk-free interest rate) are both riskless. Their returns are certain, regardless of the value of F_1. They have the same value of beta. They should yield the same rate of return. Otherwise, everyone would want to buy the security or the portfolio with the lower price (higher yields) and to sell the security or the portfolio with the higher price (lower yields).

The rate of return of the risk-free asset porfolio in (a) is equal to 0.03. The rate of return of the risk-free asset porfolio in (a) is *higher* than tje market risk-free interest rate (for example, the interest rate on treasury bill). $3\% > 1\%$. $0.03 > 0.01$. We should *short-sell* the one that *pays the lower rate of retur*n, the risk-free asset available in the market. In other words, we borrow from the market and pay the market risk-free interest rate. After that, we use the proceeds to *buy* the one that *pays the higher rate of return, risk-free portfolio in (a).* We get 3% rate of return from investing in a risk-free portfolio in (a). We would realize a profit of $(3\% - 1\%) = 2\%$. This arbitrage opportunity is inconsistent with market equilibrium.

(b) Determine the portfolio weights you need to place on A and B in order to construct a portfolio which has a unit exposure to the factor 1 (F_1). What is the expected return of the portfolio which has a unit exposure to the factor 1?

Solution.

Let w_A be the weight of asset A and w_B be the weight of asset B in a portfolio. Then, expected return of the portfolio can be written as follows.

$$\begin{aligned} E(R_p) &= aE(X) + bE(Y), \\ E(R_p) &= w_A E(R_A) + w_B E(R_B), \\ &= w_A(0.05 + 0.5F_1) + w_B(0.02 - 0.25F_1), \\ &= (0.05w_A + 0.02w_B) + (0.5w_A - 0.25w_B)F_1. \end{aligned}$$

Given that the portfolio must have beta equal to one. Therefore,

$0.5w_A - 0.25w_B = 1 \Rightarrow$ equation (1).

The sum of the weights must always be 1.

$w_A + w_B = 1 \Rightarrow$ equation (2),

Solving the system of equations (equation (1) and equation (2)) for w_A and w_B and get $w_A = \dfrac{5}{3}$ and $w_B = -\dfrac{2}{3}$.

Substitute $w_A = \dfrac{5}{3}$ and $w_B = -\dfrac{2}{3}$ in to $E(R_p) = (0.05w_A + 0.02w_B) + (0.5w_A - 0.25w_B)F_1$ and get $E(R_p) = \left(\dfrac{0.05(5) + (0.02)(-2)}{3}\right) + \left(\dfrac{0.5(5) - 0.25(-2)}{3}\right) F_1 = 0.07 + F_1$.

Hence, to construct a portfolio which has a unit exposure to the factor 1 (F_1), the weight placed on asset A must be equal to $\dfrac{5}{3}$ and the weight placed on asset B must be equal to $-\dfrac{2}{3}$ (we have to short sell asset B.). The expected return of the portfolio which has a unit exposure to the factor 1 (F_1) is then equal to $0.07 + F_1$.

2. Derive an equation for the equilibrium expected rate of return of an asset K which has $\beta_{K1} = b_K$.

Solution.

Making use $ER_i = R_f + (\lambda_1 - R_f)\beta_{i1}$, $R_f = 0.03$ (from question (a) and $E(R_p) = \lambda_1 = 0.07 + F_1$ (from question (1b) , and get

$$\begin{aligned} E(R_i) &= 0.03 + ((0.07 + F_1) - 0.03)\,\beta_{i1}, \\ &= 0.03 + (0.04 + F_1)\,\beta_{i1}, \\ &= (0.03 + 0.04\beta_{i1}) + \beta_{i1} F_1. \end{aligned}$$

The equilibrium expected rate of return of the asset K is equal to $(0.03 + 0.04b_K) + b_K F_1$.