Tentative Solution:

Quiz 2: Date: May 5, 2022 from 11.00-12.30

Question 1 (40 marks) Score.....................

Consider the Muliperiod model of consumption and portfolio choice. Let an individual in this economy has the utility function as follow:

$$\max_{C_s,\omega_s,\forall t} E_t\left[\sum_{s=t}^{T-1}\delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)+\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

Assume that there is no wage income $(y_t = 0\,\forall t)$ and a constant risk-free rate return asset , $R_{ft} = R_f$. Also asssume that n=1 and the return of a single risky asset, R_{rt},is independently and identically distributed over time. Denote the proportion of wealth invested in the risky asset at date t as ω_t.

Please read and answer the following questions carefully and completely.

Score.....................

Question 1.1 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-1, C^*_{T-1} and ω^*_{T-1}., and give an explicit expression for C^*_{T-1}

$\boxed{1}$ step 0: General form:

Let $J(W_t, t)$ denote the derived utility of wealth function.

$$J(W_t, t) \equiv \max_{C_s, \{w_s\}} E_t\left[\sum_{s=t}^{T-1} U(C_s) + B(W_T, T)\right]$$

$\boxed{2}$ step 1: start at T (Last period)

For the final period solution:

$$J(W_T, T) = E_T[B(W_T, T)] = B(W_T, T)$$

using the backwards, we can solve for the optimization problem

$\boxed{3}$ step 2: start at T-1 step back 1 period.

at the date T-1:

Let $S_t = W_t - C_t$ and Portfolio Return.

$$R_t = R_f + w_t^*(R_{it} - R_f)$$

port

From

$$J(W_{T-1}, T-1) = \max U(C_{T-1}, T-1) + E_{T-1}[B(S_{T-1} R_{T-1}, T)]$$

W_T

Expected value at date T-1

Then, we differentiate with respect to C_{T-1}, and W_{T-1} we should get:

<u>diff with C_{T-1}:</u>

$$U_C(C_{T-1}, t-1) = E_{T-1}\left[B_W(W_T, T)\, R_{T-1}\right] = 0$$

then we get

$$\cancel{\delta^{t-1}} \cdot C_{T-1}^{-\gamma} = E_{t-1}\left[\cancel{\delta^{t}}\, W_T^{-\gamma} \cdot R_{T-1}\right]$$

$$C_{T-1}^{-\gamma} = \delta E_{T-1}\left[\left[S_{T-1} R_{T-1}\right]^{-\gamma} \cdot R_{T-1}\right]$$

$$= \delta \cdot E_{T-1}\left[R_{T-1}^{1-\gamma}\right] \cdot E_{T-1}\left[(W_{T-1} - C_{T-1})^{-\gamma}\right]$$

We take power $-\frac{1}{\gamma}$ bothsides,

$$C_{T-1} = \delta^{\circ} \cdot E_{T-1}[W_{T-1} - C_{T-1}] \cdot E_{t-1}^{\circ}[R_{t-1}]$$

we move C_{T-1} to the left-HS:

$$C_{T-1} + C_{T-1} \cdot \delta^{-1/\gamma} \cdot E_{T-1}[R_{t-1}]^{-\frac{1-\gamma}{\gamma}} = \delta^{-1/\gamma} \cdot E_{T-1}[R_{t-1}]^{-\frac{1-\gamma}{\gamma}} \cdot W_{T-1}$$

$$\delta^{-1/\gamma} E_{T-1}[R_{t-1}]^{-\frac{1-\gamma}{\gamma}} = a_1 \quad \text{(define it)}$$

$$\therefore C_{T-1} = \frac{\delta^{-1/\gamma} E_{T-1}[R_{t-1}]^{-\frac{1-\gamma}{\gamma}}}{1 + \delta^{-1/\gamma} E_{T-1}[R_{t-1}]^{-\frac{1-\gamma}{\gamma}}} \cdot W_{T-1}$$

therefore

$$C_{T-1} = \left[\frac{a_1}{1+a_1}\right] \cdot W_{T-1} = C_1 \cdot W_{T-1}$$

Now, we do the first order condition with respect to the portfolio weight implies:

$$E_{T-1}[B_W(W_T, T)(R_{i,T-1} - R_f)] = 0 \quad \forall\, i = 1, 2, \ldots n$$

Therefore,

$$E_{T-1}[B_W R_{r,T-1}] = R_f \, E_{T-1}[B_W]$$

Therefore,

$$\cancel{\delta^{T}} E_{T-1}\left[(\cancel{S_{T-1}} R_{T-1})^{-\gamma} \cdot R_{r,T-1}\right] = \cancel{\delta^{T}} R_f E_{T-1}\left[(\cancel{S_{T-1}} R_{T-1})^{-\gamma}\right]$$

$$\therefore \; E_{T-1}\left[R_{T-1}^{-\gamma} \cdot R_{r,T-1}\right] = R_f E_{T-1}\left[R_{T-1}^{-\gamma}\right]$$

Port

The optimal weight is independent from the income or consumption.

$\downarrow$

Implication: everyone in this setting, has the same % invest in risky and risk-free assets.

Score....................

Question 1.2 (10 marks) Solve for the form of J(W_{T-1},T-1).

question b.

Since we know that

$$\text{from } J(W_{T-1}, T-1) = \max_{C_{T-1}, \{W_{T-1}\}} U(C_{T-1}, T-1) + E_{T-1}\left[B(S_{T-1}, T-1, T)\right]$$

we then get:

$$J(W_{T-1}, T-1) = \frac{\delta^{T-1} \cdot C_{T-1}^{*1-\gamma}}{1-\gamma} + \delta^{T} E_{T-1}\left[\frac{\left(R^{*}_{T-1}(W_{T-1} - C^{*}_{T-1})\right)^{1-\gamma}}{1-\gamma}\right]$$

$$= \delta^{T-1}\left[\frac{a_1}{1+a_1}\right]^{1-\gamma} \frac{\cdot W_{T-1}^{1-\gamma}}{1-\gamma} + \delta^{T} E_{T-1}\left[\frac{R_{T-1}^{*(1-\gamma)} \cdot W_{T-1}^{1-\gamma}}{(1-\gamma)(1+a_1)^{1-\gamma}}\right]$$

(δ left)

$T-1 \quad 1-\gamma \quad 1-\gamma$

$$= \delta \, \frac{W_{t-1}}{1-\gamma} \underbrace{\left[\left(\frac{a_1}{1+a_1} \right) + \delta E_{t-1} (R^*_{t-1})^{1-\gamma} \cdot \frac{1}{(1+a_1)^{1-\gamma}} \right]}_{\text{define it as}}$$

$$= \delta^{T-1} \cdot b_1 \cdot \frac{W_{t-1}^{1-\gamma}}{1-\gamma} \quad \#.$$

Score....................

Question 1.3 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-2, C^*_{T-2} and ω^*_{T-2}., and give an explicit expression for C^*_{T-2}

Then we can derive the first order condition for the optimal consumption level and the optimal portfolio weight at date T-2:

$$U_C(C^*_{T-2}, T-2) = E_{T-2}\left[\delta J_W(W_{T-1}, T-1) R_{T-2}\right]$$

then $\cancel{\delta^{T-2}} \cdot C_{T-2}^{-\gamma} = \cancel{\delta^{T-1}} E_{T-2}\left[b_1 W_{T-1}^{-\gamma} \cdot R_{T-2}\right]$

$$C_{T-2}^{-\gamma} = \delta E_{T-2}\left[b_1 (S_{T-1} R_{T-2})^{-\gamma} \cdot R_{T-2}\right]$$

$$C_{T-2}^{-\gamma} = \delta E_{T-2}\left[b_1 S_{T-1}^{-\gamma} \cdot R_{T-2}^{1-\gamma}\right]$$

$$C_{T-2}^{-\gamma} = \delta E_{T-2}\left[b_1 (W_{T-2} - C_{T-2})^{-\gamma} \cdot R_{T-2}^{1-\gamma} \right]$$

Therefore

$$C_{T-2}^{*} = \frac{(\delta b_1 E_{T-2}[R_{T-2}^{1-\gamma}])^{\frac{1}{\gamma}} \cdot W_{T-2}}{(1 + (\delta b_1 E_{T-2}[R_{T-2}^{1-\gamma}])^{\frac{1}{\gamma}})}$$

$$C_{T-2}^{*} = \left(\frac{a_2}{1+a_2}\right) \cdot W_{T-2}$$

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The same idea: the optimality weight:

$$E[R_{T-2}^{-\gamma} \cdot R_{r,T-2}] = R_f E[R_{T-2}^{-\gamma}]$$

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w_{T-2}^{*} does not depend on wealth or consumption as well. It is only depending on the distribution of risky asset returns (which affect the $E(\cdot)$)

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Score.....................

Question 1.4 (10 marks) Solve for the form of J(W_{T-2},T-2).Based on the pattern for T-1 and T-2, provide expressions for the optimal consumption and portfolio weight at any date T-t, t=1,2,3,....

1.4 :

for the General form, therefore,

$$c^*_{T-t} = \left(\frac{a_t}{1+a_t}\right) \cdot W_{T-t}$$

and the optimal weight,

$$E\left[R_{T-t}^{-\gamma} R_{r,T-t}\right] = R_f E\left[R_{T-t}^{-\gamma}\right]$$

$\forall\, t = 1, 2, \ldots$ #

May 7, 2022

QED.