Time Series Models

Univariate Time Series Models

- Autoregressive Integrated Moving Average (ARIMA) Models

Time-Varying Volatility Models

- Generalized Autoregressive Conditional Heteroscedasticity (GARCH) Model

ARIMA Models

Stationary Process Models

- Moving Average (MA(q))
- Autoregressive (AR(p))
- Autoregressive Moving Average (ARMA(p,q))

Nonstationary Process Models

- Autoregressive Integrated Moving Average (ARIMA(p,d,q))

Moving Average (MA(q))

Relationship between mean value of dependent variable with weighted average of random disturbances up to q period.

Moving Average process of order q or MA(q)

$$y_t = \delta + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q}$$

Autoregressive (AR(p))

Relationship between mean value of dependent variable with weighted average of its previous values up to p period.

Autoregressive process of order p or AR(p)

$$y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t$$

Autoregressive Moving Average (ARMA(p,q))

Combination of Autoregressive and Moving Average of order p and q or ARMA(p,q)

$$y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p}$$
$$+ \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q}$$

Autoregressive Integrated Moving Average (ARIMA(p,d,q))

Model for integrated nonstationary series of order d or $(I(d))$ that already transformed using combination of Autoregressive and Moving Average process of order p and q or ARIMA(p,d,q)

$$\Delta^d y_t = \delta + \phi_1 \Delta^d y_{t-1} + \phi_2 \Delta^d y_{t-2} + \cdots + \phi_p \Delta^d y_{t-p}$$
$$+\varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q}$$

Seasonality

Time Series Component

Seasonal Adjustment – Census X-12

Model with deterministic seasonals

Model with stochastic seasonals

Seasonal ARIMA (SARIMA)

Box-Jenkins Methodology

1. Identification

Identify appropriated value of p, d, &q

- use ACF, PACF, and LB Test
- use AIC or SIC – (use the model with lowest value)

$$AIC = e^{2(p+q+1)/T}\frac{\sum \hat{\varepsilon}_t^2}{T}$$

$$SIC = T^{(p+q+1)/T}\frac{\sum \hat{\varepsilon}_t^2}{T}$$

Box-Jenkins Methodology

2. Estimation

Estimate ARIMA(p,d,q) using appropriated method (e.g. NLS, MLE, or GMM)

3. Diagnostic Checking

Check whether the series follow properties of the models. e.g. check residuals from the estimated ARIMA model whether they are normally distributed.

Box-Jenkins Methodology

4. Forecasting

Choose the model that best fit the series based on your objective (e.g. accurate prediction—check forecasting error (RMSE, or Theil's Inequality Coefficient)

Model Estimation and Selection

Modelling Process

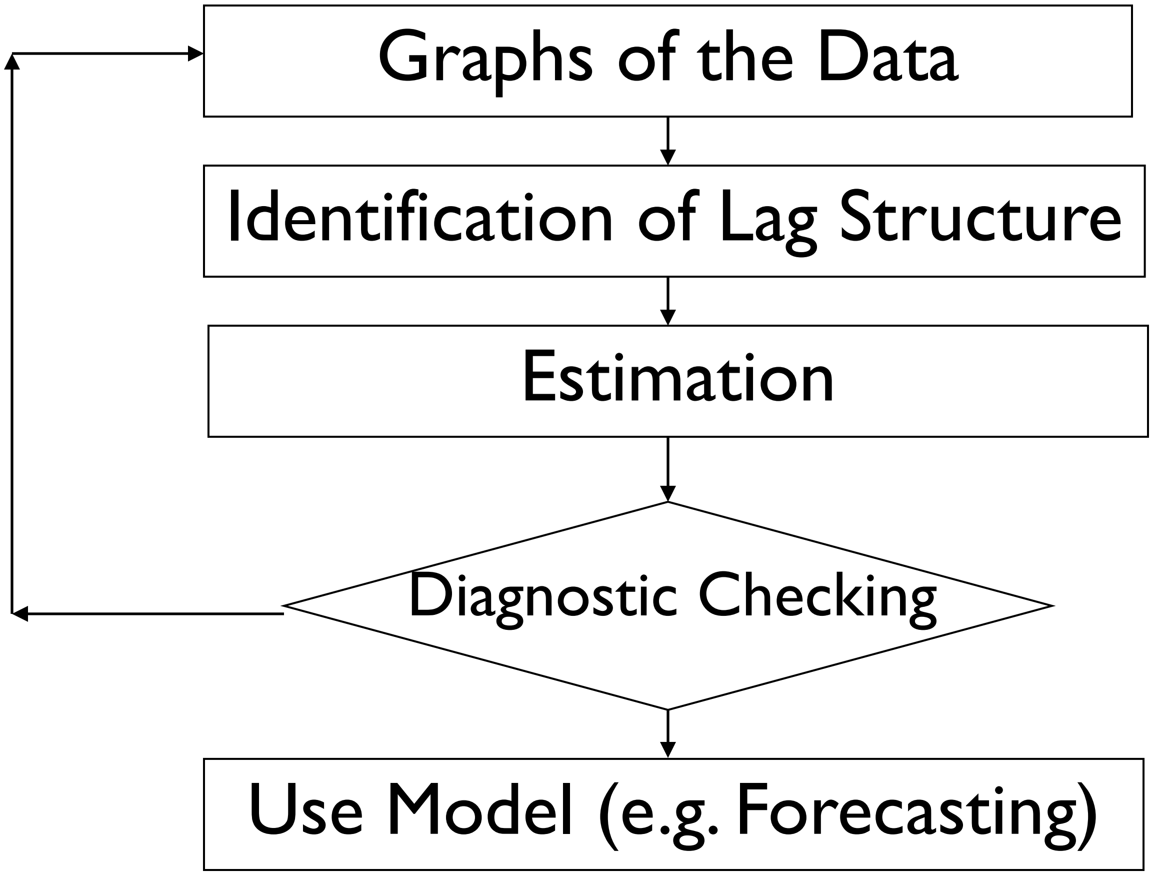
Parameter Estimation

Model Selection

Diagnostic Tests

Modelling Process

Iterative Steps in Modelling



Iterative Method

1. Graph of the data
   Time plot or Scatter plots against lagged.
2. Choice of lag structure-Use SACF or SPACF
3. Estimation of the model parameters
   Estimate ARIMA(p,d,q) using appropriated method (e.g. NLS or MLE)
4. Diagnostic Checking
   To check whether the model capture correlations in the time series.
5. Improve the model
6. Use the model-Out-of-sample forecasts

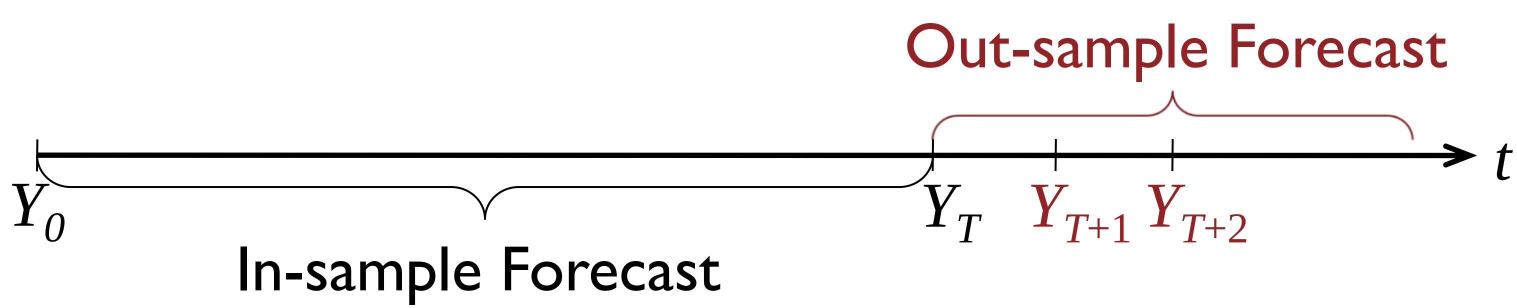
Forecasting

In-sample Forecast

Forecast that use data within the estimation sample.

Out-sample Forecast

Forecast that use data outside the estimation sample.



Forecasting

Static Forecast

Forecast that use actual value as predictors.

Dynamic Forecast

Forecast that use forecast value as predictors.

t	Y_t^A	Y_t^S	Y_t^D
1	Y_1^A		
2	Y_2^A	Y_2^S	Y_2^D
3	Y_3^A	Y_3^S	Y_3^D
4	Y_4^A	Y_4^S	Y_4^D
$\vdots$	$\vdots$	$\vdots$	$\vdots$
T-1	Y_{T-1}^A	Y_{T-1}^S	Y_{T-1}^D
T	Y_T^A	Y_T^S	Y_T^D

where: Y_t^A = Actual Value

Y_t^S = Static Forecast Value

Y_t^D = Dynamic Forecast Value

Index for Forecasting Error

Root Mean Square Error (RMSE)

$$RMSE = \sqrt{\frac{\sum_{t=1}^{T} (Y_t^A - Y_t^D)^2}{T}}$$

Theil's Inequality Coefficient

$$U = \frac{\sum_{t=1}^{T} (Y_t^A - Y_t^D)^2}{\sqrt{\left(\sum_{t=1}^{T} Y_t^A\right)^2 + \left(\sum_{t=1}^{T} Y_t^D\right)^2}}$$

Forecasting

One-step-ahead and Multi-step-ahead forecasts.

One-step-ahead $\hat{y}_{t+1} = f(Y_t)$

Multi-step-ahead $\hat{y}_{t+h} = E[y_{t+h} \mid Y_t]$

Forecasting and AR Process

One-step-ahead $\hat{y}_{t+1} = \alpha + \phi_1 y_t + \cdots + \phi_p y_{t+1-p}$

Two-step-ahead

$$\hat{y}_{t+2} = \alpha + \phi_1 \hat{y}_{t+1} + \phi_2 y_t + \cdots + \phi_p y_{t+2-p}$$

Forecasting

One-step-ahead Forecasting Error

$$y_{t+1} - \hat{y}_{t+1} = \varepsilon_{t+1}$$

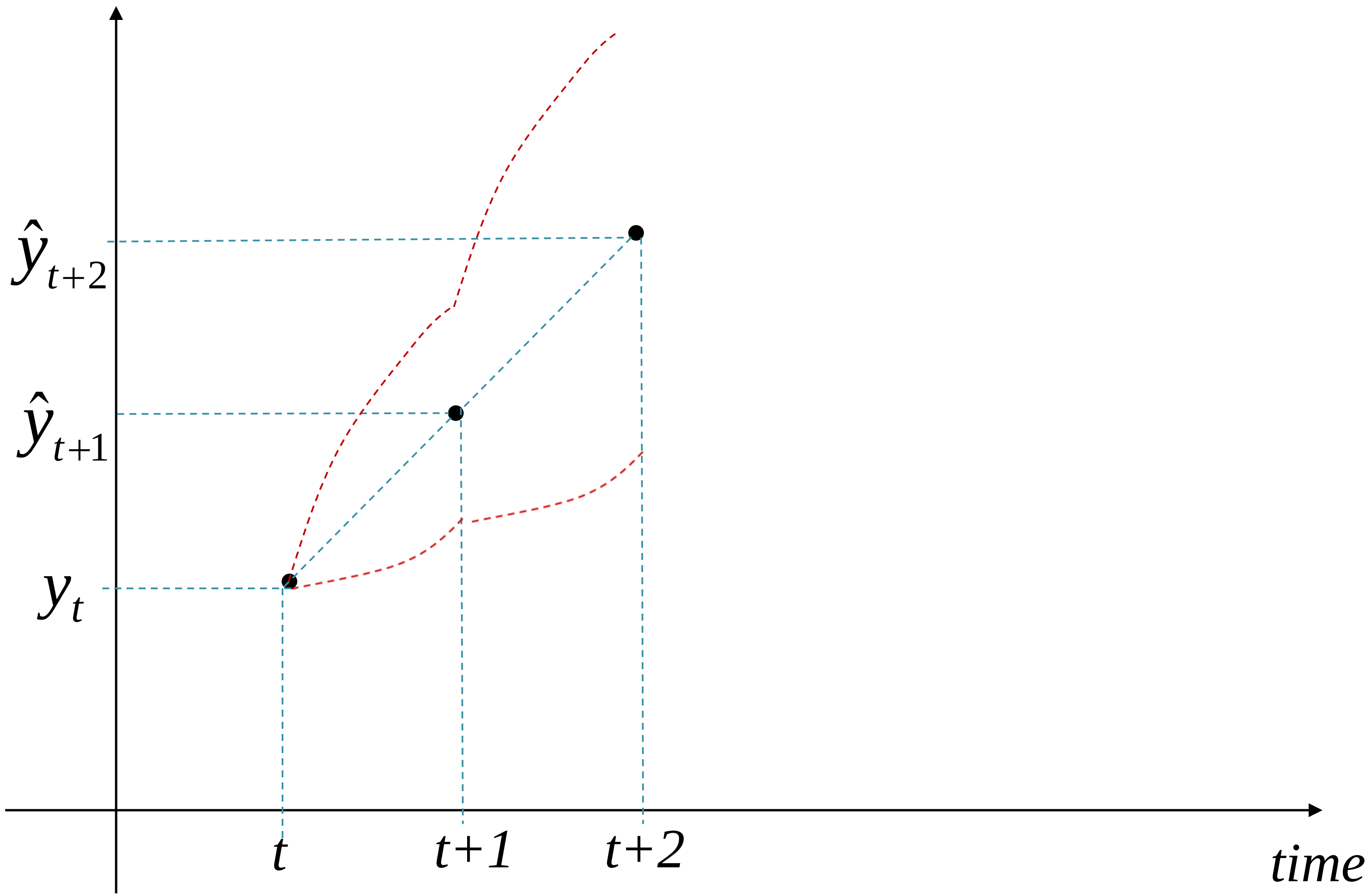
Forecasting Error Variance σ^2

Two-step-ahead Forecasting Error

$$y_{t+2} - \hat{y}_{t+2} = \varepsilon_{t+2} + \phi_1 \varepsilon_{t+1}$$

Forecasting Error Variance $\sigma^2 (1 + \phi_1^2)$

Forecasting



Non-linearities and Time-Varying Volatility

GARCH Models for Clustered Volatility

Estimation and Diagnostic Tests of GARCH

GARCH Model

Autoregressive Conditional Heteroscedasticity

ARCH(1) Model

ARCH(q) Model

ARCH in Mean (ARCH-M) Model

Generalized ARCH or GARCH(p,q) Model

Asymmetric GARCH Models

- GJR or Threshold GARCH (TARCH) Model
- Exponential GARCH (EGARCH) Model

ARCH(1) Model

$$y_t = x_t \beta + \varepsilon_t$$

$$\varepsilon_t = u_t \sqrt{\alpha_0 + \alpha_1 \varepsilon_{t-1}^2}$$

where $E[\varepsilon_t | x_t, \varepsilon_{t-1}] = 0$ then $E[\varepsilon_t | x_t] = 0$

Classical regression model follows OLS assumptions (homoscedasticity).

However, the conditional variance is:

$$Var[\varepsilon_t | \varepsilon_{t-1}] = E[\varepsilon_t^2 | \varepsilon_{t-1}] = E[u_t^2][\alpha_0 + \alpha_1 \varepsilon_{t-1}^2]$$

$$= \alpha_0 + \alpha_1 \varepsilon_{t-1}^2$$

ARCH(1) Model

Since classical regression model follows OLS assumptions, OLS estimators are efficient. However, with the conditional heteroscedasticity, nonlinear estimation method (such as MLE) can provide more efficient estimators.

Conditional log-likelihood of the model:

$$\log L = -\frac{T}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\log(\alpha_0 + \alpha_1\varepsilon_{t-1}^2) - \frac{1}{2}\sum_{t=1}^{t}\frac{\varepsilon_t^2}{\alpha_0 + \alpha_1\varepsilon_{t-1}^2}$$

where $\varepsilon_t = y_t - x_t\beta$

ARCH(q) Model

$$y_t = x_t\beta + \varepsilon_t$$

ARCH(q) process:

$$\sigma_t^2 = \alpha_0 + \alpha_1\varepsilon_{t-1}^2 + \alpha_2\varepsilon_{t-2}^2 + \cdots + \alpha_q\varepsilon_{t-q}^2$$

ARCH-in-Mean Model

$$y_t = x_t\beta + \delta\sigma_t^2 + \varepsilon_t$$

$$Var\left[\varepsilon_t \middle| \Psi_t\right] = ARCH(q)$$

Test for ARCH Effects

Hypothesis: H_0: $\alpha_1 = \alpha_2 = \ldots = \alpha_q = 0$

1. Estimate $y_t = x_t \beta + \varepsilon_t$ obtain $\hat{\varepsilon}_t$

2. Estimate

$$\hat{\varepsilon}_t^2 = \alpha_0 + \alpha_1 \hat{\varepsilon}_{t-1}^2 + \alpha_2 \hat{\varepsilon}_{t-2}^2 + \cdots + \alpha_q \hat{\varepsilon}_{t-q}^2 + u_t$$

obtain R^2

3. Calculate $TR^2 \sim \chi^2(q)$

GARCH(p,q) Model

$$y_t = x_t \beta + \varepsilon_t$$

GARCH(p,q) process:

$$\begin{aligned}\sigma_t^2 &= \alpha_0 + \delta_1 \sigma_{t-1}^2 + \delta_2 \sigma_{t-2}^2 + \cdots + \delta_p \sigma_{t-p}^2 \\ &\quad + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + \cdots + \alpha_q \varepsilon_{t-q}^2 \\ &= \alpha_0 + \sum_{j=1}^{p} \delta_j \sigma_{t-j}^2 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2\end{aligned}$$

Asymmetric GARCH Model

Glosten, Jagannathan & Runkle (GJR) or Threshold GARCH (TARCH) Model

TARCH(p,q,r) $\quad y_t = x_t\beta + \varepsilon_t$

Conditional variance process:

$$\sigma_t^2 = \alpha_0 + \sum_{j=1}^{p} \delta_j \sigma_{t-j}^2 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{k=1}^{r} \gamma_k \varepsilon_{t-k}^2 I_{t-k}$$

$$\text{where} \quad I_{t-1} = 1 \text{ if } \quad \varepsilon_{t-1} < 0$$
$$= 0 \text{ otherwise}$$

Asymmetric GARCH Model

Exponential GARCH (EGARCH) Model

EGARCH(p,q,r) $y_t = x_t\beta + \varepsilon_t$

Conditional variance process:

$$\ln(\sigma_t^2) = \alpha_0 + \sum_{j=1}^{p} \delta_j \ln(\sigma_{t-j}^2) + \sum_{i=1}^{q} \alpha_i \left| \frac{\varepsilon_{t-i}}{\sigma_{t-i}} \right| + \sum_{k=1}^{r} \gamma_k \frac{\varepsilon_{t-k}}{\sigma_{t-k}}$$