Chapter 8, Ex.7

Given
$$\frac{xz^2}{x+y} + y^2 = 0$$

Rearrange so that can diff both sides easily

$$xz^2 = -y^2(x+y)$$

$$xz^2 = -y^2x - y^3$$

$$\text{LHS} = \text{RHS}$$

Partial differentiate both sides w.r.t. x (treat y as a constant)

Apply product rule to LHS

$$(1)z^2 + x\left(2z\frac{\partial z}{\partial x}\right) = -y^2$$

$$2zx\left(\frac{\partial z}{\partial x}\right) = -y^2 - z^2$$

$$\frac{\partial z}{\partial x} = \frac{-(y^2+z^2)}{2zx}$$

when $x=-1$, $y=2$, $z=2$

$$\frac{\partial z}{\partial x} = \frac{-(4+4)}{2(2)(-1)} = \frac{-8}{-4} = 2$$

Ex. 8

$$\text{LHS} = \text{RHS}$$

$$e^{yz} = -xyz$$

Partial differentiate both sides w.r.t. x (treat y as a constant)

For L.HS, write LHS $= e^u$ where $u = yz$

↳ treat this as a constant when diff partially w.r.t. x

$$\frac{\partial \text{LHS}}{\partial x} = e^u \frac{\partial u}{\partial x}$$

$$= e^u\left(y\frac{\partial z}{\partial x}\right)$$

$$= \left(ye^{yz}\right)\left(\frac{\partial z}{\partial x}\right)$$

For R.H.S, use product rule and treat y as a constant

$$\frac{\partial \text{RHS}}{\partial x} = -(1)yz + (-xy)\frac{\partial z}{\partial x} = -yz - xy\frac{\partial z}{\partial x}$$

$$\therefore \quad ye^{yz}\left(\frac{\partial z}{\partial x}\right) = -yz - xy\left(\frac{\partial z}{\partial x}\right)$$

$$\left[ye^{yz} + xy\right]\left(\frac{\partial z}{\partial x}\right) = -yz$$

Ex. 8 (cont.)

$$\frac{\partial z}{\partial x} = \frac{-yz}{ye^{yz} + xy}$$

when $x = -\frac{e^2}{2}$, $y = 1$, $z = 2$

$$\frac{\partial z}{\partial x} = \frac{-(1)(2)}{(1)e^{(1)(2)} + \left(-\frac{e^2}{2}\right)(1)} = \frac{-2}{e^2 - \frac{e^2}{2}}$$

$$= \frac{-2}{\frac{e^2}{2}} = \frac{-4}{e^2}$$

Ex. 9

$$se^{r^2+u^2} = u\ln(t^2+1)$$

LHS = RHS

Partial differentiate both sides with respect to u (treat r as a constant)

$$LHS = se^{r^2+u^2}$$

$$= se^m \quad \text{where } m = r^2+u^2 \qquad \frac{\partial m}{\partial u} = \frac{\text{diff}\,(\text{const} + u^2)}{du} = 2u$$

$$\frac{\partial LHS}{\partial u} = se^m \frac{\partial m}{\partial u}$$

$$= se^{r^2+u^2}(2u)$$

$$RHS = u\ln(t^2+1)$$

$$= u\ln m \quad \text{where } m = t^2+1$$

using product rule $\Rightarrow$ $RHS = (1)\ln m + u\frac{1}{m}\frac{dm}{du}$

$$= \ln(t^2+1) + \frac{u}{t^2+1}\left(2t\frac{\partial t}{\partial u}\right)$$

$$\therefore\ se^{r^2+u^2}(2u) = \ln(t^2+1) + \frac{u}{t^2+1}\left(2t\frac{\partial t}{\partial u}\right)$$

$$\frac{u}{t^2+1}\cdot 2t\left(\frac{\partial t}{\partial u}\right) = 2use^{r^2+u^2} - \ln(t^2+1)$$

$$\left(\frac{\partial t}{\partial u}\right) = \frac{(t^2+1)\{2sue^{r^2+u^2} - \ln(t^2+1)\}}{2ut}$$

Chapter 8, Ex. 10

$$f(x,y) = 7x^2 + 3y$$

$$\frac{\partial f}{\partial x} = 14x \xRightarrow{\text{diff with } x} \frac{\partial^2 f}{\partial x^2} = 14$$

$$\xRightarrow{\text{diff w.r.t. } y} \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$\frac{\partial f}{\partial y} = 3 \xRightarrow{\text{diff w.r.t. } y} \frac{\partial^2 f}{\partial y^2} = 0$$

$$\xRightarrow{\text{diff w.r.t. } x} \frac{\partial^2 f}{\partial y \partial x} = 0$$

Ex. 11

$$f(x,y) = x^2 y + x^2 y^2$$

$$\frac{\partial f}{\partial x} = 2xy + 2xy^2 \xRightarrow{\text{further diff w.r.t. } x} \frac{\partial^2 f}{\partial x^2} = 2y + 2y^2$$

$\Downarrow$ partial diff w.r.t. y

$$\frac{\partial^2 f}{\partial x \partial y} = 2x + 4xy$$

$$\frac{\partial f}{\partial y} = x^2 + x^2(2y) \Rightarrow \frac{\partial^2 f}{\partial y^2} = x^2(2) = 2x^2$$

$\Downarrow$

$$\frac{\partial^2 f}{\partial y \partial x} = 2x + 2x(2y) = 2x + 4xy$$

Ex.12 $f(x,y) = (x+y)^2(xy)$ ⇒ you may keep this as this and use product rule later but since this is not difficult to multiply, I choose to expand it out for less complication when trying to partial diff later!

$= (x^2+2xy+y^2)(xy)$

$= x^3y + 2x^2y^2 + xy^3$

$\frac{\partial f}{\partial x} = 3x^2y + 4xy^2 + y^3 \Rightarrow \frac{\partial^2 f}{\partial x^2} = 6xy + 4y^2 + 0$

$\Downarrow$

$\frac{\partial^2 f}{\partial x \partial y} = 3x^2 + 8xy + 3y^2 \xRightarrow[y]{\text{diff w.r.t}} \frac{\partial^3 f}{\partial x \partial y^2} = 8x + 6y$

$\frac{\partial f}{\partial y} = x^3 + 2x^2(2y) + x3y^2$

$= x^3 + 4x^2y + 3xy^2 \Longrightarrow \frac{\partial^2 f}{\partial y^2} = 4x^2 + 3x(2y)$

$= 4x^2 + 6xy$

$\Downarrow$ partial diff w.r.t. x

$\frac{\partial^3 f}{\partial y^2 \partial x} = 8x + 6y$

$\Downarrow$

$\frac{\partial^2 f}{\partial y \partial x} = 3x^2 + 8xy + 3y^2$

Ex.13 $f(x,y) = y^2e^x + \ln(xy)$

$\frac{\partial f}{\partial x} = y^2e^x + \frac{1}{xy}\frac{\partial(xy)}{\partial x} = y^2e^x + \frac{1}{xy} \cdot y$

$= y^2e^x + \frac{1}{x}$

partial diff w.r.t. y $\Downarrow$

$\frac{\partial^2 f}{\partial x \partial y} = e^x(2y)$

partial diff w.r.t. y again $\Downarrow$

$\frac{\partial^3 f}{\partial x \partial y^2} = e^x(2) = 2e^x$

when $x=1, y=1$

$\frac{\partial^3 f}{\partial x \partial y^2} = 2e$

Chapter 8, Ex. 14

$$z = 2x^4 + 3x^3y^3 + xy^2 + y$$

$$\frac{\partial z}{\partial x} = 8x^3 + 9x^2y^3 + y^2$$

⇓ diff this w.r.t. y, treating x as a constant

$$\frac{\partial^2 z}{\partial x \partial y} = 9x^2(3y^2) + 2y$$

$$= 27x^2y^2 + 2y$$

$$\frac{\partial z}{\partial y} = 3x^3(3y^2) + x(2y) + (1)$$

$$= 9x^3y^2 + 2xy + 1$$

⇓ diff this w.r.t. x, treating y as a constant

$$\frac{\partial^2 z}{\partial y \partial x} = 9y^2(3x^2) + 2y(1)$$

$$= 27x^2y^2 + 2y$$

$$\therefore \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = 27x^2y^2 + 2y$$

Ex. 15 Find the rate of change of the total cost (C) w.r.t. price of digital camera (P_C) ⇒ This is to find $\frac{\partial C}{\partial P_C}$ while treating P_F as a constant

Given $C = 30q_C + 0.015q_Cq_F + q_F + 900$

$$\frac{\partial C}{\partial P_C} = 30\frac{\partial q_C}{\partial P_C} + 0.015\frac{\partial q_C}{\partial P_C} \cdot q_F + 0.015q_C \cdot \frac{\partial q_F}{\partial P_C} + \frac{\partial q_F}{\partial P_C}$$

Given $q_C = 9000P_C^{-1}P_F^{-1/2}$ $\qquad q_F = 2000 - P_C - 400P_F$

$$\frac{\partial q_C}{\partial P_C} = 9000P_F^{-1/2}(-1)P_C^{-2} = -\frac{9000}{\sqrt{P_F}} \cdot \frac{1}{P_C^2}$$

$$\frac{\partial q_F}{\partial P_C} = -1$$

$$\therefore \frac{\partial C}{\partial P_C} = \frac{30(-9000)}{P_C^2\sqrt{P_F}} + 0.015(2000 - P_C - 400P_F)\left[\frac{-9000}{P_C^2\sqrt{P_F}}\right] + \frac{0.015 \times 9000}{P_C\sqrt{P_F}} \cdot (-1) + (-1)$$

Ex. 17

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s}$$

$$= [3(2x)y + yz](-3) + [3x^2(1) + xz(1) - 4(2y)z^3](1) + [xy(1) - 4y^2(3z^2)](-1)$$

$$= (6xy+yz)(-3) + (3x^2+xz-8yz^3) - (xy-12y^2z^2)$$

$$= -18xy - 3yz + 3x^2 + xz - 8yz^3 - xy + 12y^2z^2$$

$$= -19xy - 3yz + 3x^2 + xz - 8yz^3 + 12y^2z^2$$

$$= 3x^2 - 19xy + xz - 3yz - 8yz^3 + 12y^2z^2$$

$$= x(3x - 19y + z) + yz(-3 - 8z^2 + 12yz)$$

Ex. 18

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$z = \frac{x}{y} + \frac{e^y}{y} = xy^{-1} + e^y \cdot y^{-1}$$

$$\frac{\partial z}{\partial x} = (1)(y^{-1}) + 0 = \frac{1}{y}$$

$$\frac{\partial z}{\partial y} = x(-y^{-2}) + e^y(y^{-1}) + e^y(-y^{-2})$$

$$= \frac{-x}{y^2} + \frac{e^y}{y} - \frac{e^y}{y^2}$$

$x = rs + se^{rt}$ $\quad$ $y = 9 + rt$

$$\frac{\partial x}{\partial s} = r + e^{rt}$$ e.g. rt (treat other variables except s as constants)

$$\frac{\partial y}{\partial s} = 0$$

$$\therefore \frac{\partial z}{\partial s} = \frac{1}{y}(r + e^{rt})$$

when $r = -2, s = 5, t = 4$ $\Rightarrow \frac{\partial z}{\partial s} = \frac{1}{9 + (-2)(4)}\left(-2 + e^{(-2)(4)}\right) = \frac{-2 + e^{-8}}{1}$

Ex. 19 $\quad y = x^2 \ln(x^4+6) \qquad x = (r+3s)^6$

$$\frac{\partial y}{\partial r} = \frac{\partial y}{\partial x} \cdot \frac{\partial x}{\partial r}$$

$$\frac{\partial y}{\partial x} = x^2\left[\frac{1}{x^4+6} \cdot 4x^3\right] + 2x \ln(x^4+6)$$ product rule

$$= \frac{4x^5}{x^4+6} + 2x\ln(x^4+6)$$

$$\frac{\partial x}{\partial r} = \frac{\partial (r+3s)^6}{\partial r} = \frac{\partial m^6}{\partial r} = 6m^5 \frac{\partial m}{\partial r} \quad \text{where } m = r+3s$$

$$= 6(r+3s)^5(1) = 6(r+3s)^5$$

$$\therefore \frac{\partial y}{\partial r} = \left[\frac{4x^5}{x^4+6} + 2x\ln(x^4+6)\right] 6(r+3s)^5$$

$$= 12x(r+3s)^5\left[\frac{2x^4}{x^4+6} + \ln(x^4+6)\right]$$

Ex. 20 $$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$= \left(y e^{xy}\right)(1) + \left(x e^{xy}\right)(1)$$

$$= (x+y)e^{xy}$$

given $x = r-4s$, $y = r-s$

$$\therefore \frac{\partial z}{\partial r} = (r-4s+r-s)\, e^{(r-4s)(r-s)}$$

$$= (2r-5s)\, e^{r^2-5sr+4s^2}$$