Assignment 4

DUE DATE: Tuesday 9^{nd}, March 2021.

I pledge to the Honor Code and to obey all rules for taking and performing homework assignments as specified by the course instructor.

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Question 1 (50 points) Your score......................

Given the daily log returns : (R_t) can be explained by the AR(2) model as following:

$$(1 - 1.5B + 0.9B^2)R_t = 0.25 + \varepsilon_t$$

where ε_t is distributed as the Gaussian White Noise with mean (μ) = 0 and variance (σ^2) =0.25
B lag-operator

$\gamma_1 < cov(r_t, r_{t-1})$

Question 1.1 (10 points) Your score......................

From the above AR(2) model, Is the model weakly stationary? Write down the reverse characteristic equation and find out the conditions to support your answer.

- Characteristic equation
  $1 - 1.5B + 0.9B^2$
- Reverse characteristic equation

$B^2 - 1.5B + 0.9 = 0$

AR(p) model is weakly stationary if all of reversed characteristic root value are less than 1 in modulus

Since this is 2^{nd} degree of polynomial we have two roots.

$$B = \frac{-(-1.5) \pm \sqrt{(-1.5)^2 - 4(1)(0.9)}}{2(1)}$$

$B = 0.75 \pm i\sqrt{0.3375}$

$|B| = 0.9486 < 1$ #

Since, reversed characteristic roots fall inside unit circle, we can say that the model is stationarity

Question 1.2 (10 points) Your score......................

Calculate the unconditional mean: $E(R_t)$ of R_t and the conditional mean: $E(R_t|F_{t-1})$

$(1-1.5B+0.9B^2)R_t = 0.25+\varepsilon_t$

$R_t - 1.5BR_t + 0.9B^2R_t = 0.25+\varepsilon_t$, B is lag term $BR_t = R_{t-1}$

Unconditional mean

$R_t = 1.5R_{t-1} - 0.9R_{t-2} + 0.25 + \varepsilon_t$

$E(R_t) = 1.5E(R_{t-1}) - 0.9E(R_{t-2}) + E(0.25) + E(\varepsilon_t)$

under stationary condition, $E(R_t) = E(R_{t-1}) = E(R_{t-2}) = \mu$, and ε_t is gaussian white noise

$E(R_t) - 1.5E(R_t) + 0.9E(R_t) = 0.25 + 0$

$E(R_t) = \frac{0.25}{(1-1.5+0.9)} = \frac{0.25}{0.4} = 0.625$ #

Conditional mean

$E(R_t|R_1, R_2, \ldots, R_{t-1}) = E(R_t|\cdot) = E(R_t|F_{t-1})$

$E(R_t|\cdot) = 1.5E(R_{t-1}|\cdot) - 0.9E(R_{t-2}|\cdot) + 0.25 + 0$

$= 0.25 + 1.5R_{t-1} - 0.9R_{t-2}$ #

Question 1.3 (10 points) Your score......................

Find out the unconditional variance: $Var(R_t)$ of R_t and conditional variance $Var(R_t|F_{t-1})$ of R_t

Unconditional variance

$$R_t = 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + \varepsilon_t$$

under stationarity condition, $E(R_t) = E(R_{t-1}) = E(R_{t-2}) = \mu$

$$R_t = \phi_0 + \phi_1 R_{t-1} + \phi_2 R_{t-2} + \varepsilon_t$$

From, $\phi_0 = E(R_t)(1-\phi_1-\phi_2)$

$$R_t = \mu - \mu\phi_1 - \mu\phi_2 + \phi_1 R_{t-1} + \phi_2 R_{t-2} + \varepsilon_t$$

$$R_t - \mu = \phi_1(R_{t-1}-\mu) + \phi_2(R_{t-2}-\mu) + \varepsilon_t$$

$$(R_t-\mu)^2 = \phi_1^2(R_{t-1}-\mu)^2 + \phi_2^2(R_{t-2}-\mu)^2 + 2\phi_1\phi_2(R_{t-1}-\mu)(R_{t-2}-\mu) + \varepsilon_t^2 + 2\left(\phi_1(R_{t-1}-\mu) + \phi_2(R_{t-2}-\mu)\right)(\varepsilon_t)$$

$$E(R_t-\mu)^2 = \phi_1^2 E(R_{t-1}-\mu)^2 + \phi_2^2 E(R_{t-2}-\mu)^2 + 2\phi_1\phi_2 E((R_{t-1}-\mu)(R_{t-2}-\mu)) E(\varepsilon_t - 0)^2 + 2\left(\phi_1 E(R_{t-1}-\mu)(\varepsilon_t) + \phi_2 E(R_{t-2}-\mu)(\varepsilon_t)\right)$$

under stationarity condition $var(r_t) = var(r_{t-1}) = \dots = \text{constant}$

$$var(R_t) = \phi_1^2 var(R_t) + \phi_2^2 var(R_t) + 2\phi_1\phi_2\gamma_1 + var(\varepsilon_t) + 0$$

$$var(R_t)\left[1-\phi_1^2-\phi_2^2\right] = \frac{\sigma_\varepsilon^2 + 2\phi_1\phi_2\gamma_1}{1-\phi_1^2-\phi_2^2} = \frac{0.25 + 2(1.5)(-0.9)(\gamma_1)}{1-(1.5)^2-(-0.9)^2}$$

$$= \frac{0.25 - 2.7\gamma_1}{-2.06} \#$$

Conditional variance

$$Var(r_t|\cdot) = \sigma_\varepsilon^2 = 0.25 \#$$

, we can't go back to change the past

Question 1.4 (10 points)

Your score......................

Calculate the autocorrelation: ρ_l for l=1 and 2 of R_t. Also, write down the autocorrelation: ρ_l when l $\geq$ 2.

ρ_1 and ρ_2

$$R_t = \phi_0 + \phi_1 (R_{t-1}) + \phi_2 (R_{t-2}) + \varepsilon_t$$

under stationarity cond.

we know that $\phi_0 = E(r_t)[1 - \phi_1 - \phi_2]$,

$$R_t - \mu = \phi_1 (R_{t-1} - \mu) + \phi_2 (R_{t-2} - \mu) + \varepsilon_t$$

$$(R_t - \mu)(R_{t-l} - \mu) = \phi_1 (R_{t-1} - \mu)(R_{t-l} - \mu) + \phi_2 (R_{t-2} - \mu)(R_{t-l} - \mu) + \varepsilon_t (R_{t-l} - \mu)$$

Take $E(\cdot)$ in to the equation

$$Cov(r_t, r_{t-l}) = \phi_1 Cov(R_{t-1}, R_{t-l}) + \phi_2 cov(R_{t-2}, R_{t-l}) + 0$$

divide the equation above with γ_0

$$\rho_l = \phi_1 \rho_{l-1} + \phi_2 \rho_{l-2} \quad \text{for } l > 0$$

For lag-1 ACF

$$\rho_1 = \phi_1 + \phi_2 \rho_1$$

$$\rho_1 = \frac{\phi_1}{1 - \phi_2} \quad \text{for } l = 1$$

$$\rho_l = \phi_1 \rho_{l-1} + \phi_2 \rho_{l-2} \quad \text{for } l \geqslant 2$$

Question 1.5 (10 points)

Your score......................

Given $R_{1000} = 0.01$ $R_{999} = 0.02$ $R_{998} = 0.03$ $\varepsilon_{1000} = -0.01$ $\varepsilon_{999} = -0.02$ $\varepsilon_{998} = -0.03$ Obtain 1-step, 2-setp 95 % interval forecasts for R_t at the forecast origin t = 1000. Also the ∞-step 95 % interval forecasts for R_t. Draw these intervals.

Forecast 1-step

$R_t = 0.25 + 1.5 R_{t-1} - 0.9 R_{t-2} + \varepsilon_t$

Forecasting

1-step ,

$R_{1001} = 0.25 + 1.5 R_{t-1} - 0.9 R_{t-2} + a_{1001}$

$\hat{R}_{1001} = 0.25 + 1.5 R_{t-1} - 0.9 R_{t-2}$

$Y_{n+1} - \hat{Y}_{n+1} = a_{n+1}$

$var(Y_{n+1} - \hat{Y}_{n+1}) = \sigma^2$

$\hat{R}_{1001} = 0.25 + 1.5(0.01) - 0.9(0.02) = 0.247$

2-step : $\hat{R}_{1002} = 0.25 + 1.5(0.247) - 0.9(0.01) = .6025$

∞-step : $y_{T+k} \to \mu = \frac{\phi_0}{1-\phi_1-\phi_2}$ As $k \to \infty$

1 Step

$\sigma = 0.25$

95% CI

$0.247 \pm 2(0.25)$

2 step

$\sigma = 0.25(1+\phi_1^2)^{1/2}$

$= 0.25(1+1.5^2)^{1/2}$

$= 0.451$

95% CI

$0.6025 \pm 2(0.451)$