Question 1:

Given the equation for the production function

$$\frac{18}{(0.2K^{-0.4}+0.8L^{-0.4})^{-2.5}}$$

$$Q = f(K, L) = 18 * [0.2K^{-0.4} + 0.8L^{-0.4}]^{-2.5}$$

1.1 What type of constant return to scale does the production function exhibit?
1.2 Is the production function increasing with respect to K and L?
1.3 Use the implicit function rule to find the marginal rate of technical substitution (MRTS) of L for K.
1.4 Use the Hessian matrix. Proof that the production function is concave.

1.1) $K_0, L_0 \ni Q_0 = 18(0.2K_0^{-0.4} + 0.8L_0^{-0.4})^{-2.5}$

$tK_0, tL_0 \ni Q = 18(0.2(tK_0)^{-0.4} + 0.8(tK_0)^{-0.4})$

$Q = (t^{-0.4})^{-2.5}\ 18[0.2K_0^{-0.4} + 0.8K_0^{-0.4}]^{-2.5}$

$Q = t^{1} \cdot Q_0$

$\because$ Degree of H.M. $= 1$ $\therefore$ f is constant return to scale.

1.2) Use chain rule

$Q = 18 \cdot (0.2K^{-0.4} + 0.8L^{-0.4})^{-2.5}$

1] $\frac{\partial Q}{\partial K} = (-45)(0.2K^{-0.4} + 0.8L^{-0.4})^{-3.5}(-0.08K^{-1.4}) > 0$

2] $\frac{\partial Q}{\partial L} = (-45)(0.2K^{-0.4} + 0.8L^{-0.4})^{-3.5}(-0.32L^{-1.4}) > 0$

$\therefore$ production function is increasing function

1.3) $f(k,L)=0$

$$f(K,L) = 18\cdot(0.2K^{-0.4}+0.8L^{-0.4})^{-2.5}=0$$

$$MRTS = -\frac{f_L}{f_K} = \frac{18(-2.5)[0.2K^{-0.4}+0.8L^{-0.4}]^{-3.5}(-0.32L^{-1.4})}{18(-2.5)[0.2K^{-0.4}+0.8L^{-0.4}]^{-3.5}(-0.08K^{-1.4})}$$

$$\frac{MP_L}{MP_K} = \frac{4L^{-1.4}}{K^{-1.4}} \;\star$$

1.4) $H = \begin{bmatrix} f_{KK} & f_{KL} \\ f_{LK} & f_{LL} \end{bmatrix}$

- $f_{KK} = \frac{\partial f}{\partial K} = (3.6K^{-1.4})(0.2K^{-0.4}+0.8L^{-0.4})^{-3.5}$

$$\frac{\partial f_K}{\partial K} = (3.6K^{-1.4})(3.5)[0.2K^{-0.4}+0.8L^{-0.4}]^{-4.5}(-0.08K^{-1.4}) + (0.2K^{-0.4}+0.8L^{-0.4})^{-3.5}(-5.04K^{-2.4})$$
$$= (1.008K^{-2.8})[0.2K^{-0.4}+0.8L^{-0.4}]^{-4.5} + [0.2K^{-0.4}+0.8L^{-0.4}]^{-3.5}(-5.04K^{-2.4})$$

- $f_{KL} = \frac{\partial f}{\partial K} = (3.6K^{-1.4})(0.2K^{-0.4}+0.8L^{-0.4})^{-3.5}$

$$\frac{\partial f_K}{\partial L} = (3.6K^{-1.4})(3.5)(0.2K^{-0.4}+0.8L^{-0.4})^{-4.5}(-0.32L^{-1.4}) + (0.2K^{-0.4}+0.8L^{-0.4})^{-3.5}(0)$$
$$= (4.032K^{-1.4}L^{-1.4})(0.2K^{-0.4}+0.8L^{-0.4})^{-4.5}$$

- $f_{LK} = \frac{\partial f}{\partial L} = (14.4L^{-1.4})(0.2K^{-0.4}+0.8L^{-0.4})^{-3.5}$

$$\frac{\partial f_L}{\partial K} = (14.4L^{-1.4})(3.5)(0.2K^{-0.4}+0.8L^{-0.4})^{-4.5}(-0.08K^{-1.4}) + (0.2K^{-0.4}+0.8L^{-0.4})^{-3.5}(0).$$
$$= (4.032K^{-1.4}L^{-1.4})(0.2K^{-0.4}+0.8L^{-0.4})^{-4.5}$$

- $f_{LL} = \frac{\partial f}{\partial L} = (14.4L^{-1.4})(0.2K^{-0.4}+0.8L^{-0.4})^{-3.5}$

$$\frac{\partial f_L}{\partial L} = (14.4L^{-1.4})(3.5)(0.2K^{-0.4}+0.8L^{-0.4})^{-4.5}(-0.32L^{-1.4}) + (0.2K^{-0.4}+0.8L^{-0.4})^{-3.5}(-20.16L^{-2.4})$$
$$= (16.128L^{-2.8})(0.2K^{-0.4}+0.8L^{-0.4})^{-4.5} + (0.2K^{-0.4}+0.8L^{-0.4})^{-3.5}(-20.16L^{-2.4})$$

$$|H_1| = |(1.008K^{-2.8})(0.2K^{-0.4}+0.8L^{-0.4})^{-4.5} + (0.2K^{-0.4}+0.8L^{-0.4})^{-3.5}(-5.04K^{-2.4})|$$
$$= (1.008^{-2.8})(0.2K^{-0.4}+0.8L^{-0.4})^{-4.5} + (0.2K^{-0.4}+0.8L^{-0.4})^{-3.5}(-5.04K^{-2.4})$$

$$|H_2| = \begin{bmatrix} f_{KK} & f_{KL} \\ f_{LK} & f_{LL} \end{bmatrix}$$

$$= \Big(\{(1.008K^{-2.8})(0.2K^{-0.4}+0.8L^{-0.4})^{-4.5} + (0.2K^{-0.4}+0.8L^{-0.4})^{-3.5}(-5.04K^{-2.4})\} \times \{(16.128L^{-2.8})(0.2K^{-0.4}+0.8L^{-0.4})^{-4.5} + (0.2K^{-0.4}+0.8L^{-0.4})^{-3.5}(-20.16L^{-2.4})\} \Big]$$
$$- \Big[\{(4.032K^{-1.4}L^{-1.4})(0.2K^{-0.4}+0.8L^{-0.4})^{-4.5}\} \times \{(4.032K^{-1.4}L^{-1.4})(0.2K^{-0.4}+0.8L^{-0.4})^{-4.5}\} \Big]$$

$\therefore$ $|H_1| < 0$; $\forall K, \forall L$

$|H_2| > 0$; $\forall K, \forall L$

$\therefore$ H is negative definite due to $d^2 q < 0$. So, the production function is concave.

the number of labor employed, respectively. Assume that the unit price of K and L are set equal to "r" and "w", respectively. Consider the following problems.

3.1) What type of the return to scale technology does the production function exhibit?

From now on, assume that c = ¼ . Consider the following problems.

3.2) Construct the profit function of the monopolist. (Hint: your profit function should be expressed in terms of K and L.)

3.3) The firm wants to maximize profit and seek for combination of the two factor inputs . Derive the demand for factor inputs, capital and labor.

3.4) How does the demand for labor vary with respect to w and r? Show your result by using partial derivative.

3.5) Confirm your answer with the second-order condition.

3.1

$$= (tK)^{\frac{1}{3}}(tL)^{\frac{2}{3}}$$

$$= t\,(K)^{\frac{1}{3}}(L)^{\frac{2}{3}}$$

constant return to scale

3.2

$$\Pi = TR - TC$$

$$P = Q^{-\frac{1}{4}}$$

$$Q = K^{\frac{1}{3}}L^{\frac{2}{3}}$$

$$= P \cdot Q - C(Q)$$

$$= (Q^{-\frac{1}{4}})(Q) - (wL + rK)$$

$$= (Q^{\frac{3}{4}}) - (wL + rK)$$

$$= (K^{\frac{1}{3}}L^{\frac{2}{3}})^{\frac{3}{4}} - wL - rK$$

$$= K^{\frac{1}{4}}L^{\frac{1}{2}} - wL - rK$$

3.3

$$\frac{\partial \Pi}{\partial K} = \frac{1}{4}K^{-\frac{3}{4}}L^{\frac{1}{2}} - r$$

$$\frac{\partial \Pi}{\partial L} = \frac{1}{2}K^{\frac{1}{4}}L^{-\frac{1}{2}} - w$$

$$r = \frac{1}{4}K^{-\frac{3}{4}}L^{\frac{1}{2}}$$

$$(K^{\frac{3}{4}})^{\frac{4}{3}} = (\frac{1}{4}r^{-1}L^{\frac{1}{2}})^{\frac{4}{3}}$$

$$\boxed{K^* = (\frac{1}{4}r^{-1}L^{\frac{1}{2}})^{\frac{4}{3}}}$$

$$w = \frac{1}{2}K^{\frac{1}{4}}L^{-\frac{1}{2}}$$

$$(L^{\frac{1}{2}})^2 = (\frac{1}{2}K^{\frac{1}{4}}w^{-1})^2$$

$$\boxed{L^* = (\frac{1}{2}K^{\frac{1}{4}}w^{-1})^2}$$

3.4

$L^* = \left(\frac{1}{2} k^{\frac{1}{4}} w^{-1}\right)^2 = L^* = \frac{1}{4} k^{\frac{1}{2}} w^{-2}$

$K^* = \left(\frac{1}{4} r^{-1} L^{\frac{1}{2}}\right)^{\frac{4}{3}}$

Substitude k into L^*

$L^* = \frac{1}{4}\left(\frac{1}{4} r^{-\frac{4}{3}} L^{\frac{4}{6}}\right)^{\frac{1}{2}} w^{-2}$

$= \frac{1}{8} r^{-\frac{4}{6}} L^{\frac{4}{12}} w^{-2}$

$= \frac{1}{8} r^{-\frac{2}{3}} L^{\frac{1}{3}} w^{-2}$

$\frac{\partial L^*}{\partial w} = -\frac{2}{4} k^{\frac{1}{2}} w^{-3} = -\frac{1}{2}\frac{k^{\frac{1}{2}}}{w^3}$ $\quad w\uparrow \Rightarrow L^*\downarrow$

$\frac{\partial L^*}{\partial r} = -\frac{1}{12} r^{-\frac{5}{3}} L^{\frac{1}{3}} w^{-2}$ $\quad r\uparrow \Rightarrow L^*\downarrow$

3.5

$$\begin{bmatrix} \pi_{kk} & \pi_{kL} \\ \pi_{LL} & \pi_{Lk} \end{bmatrix} = \begin{bmatrix} -\frac{3}{16} k^{-\frac{7}{4}} L^{\frac{1}{2}} & \frac{1}{8} k^{-\frac{3}{4}} L^{-\frac{1}{2}} \\ \frac{1}{8} k^{-\frac{3}{4}} L^{-\frac{1}{2}} & -\frac{1}{4} k^{\frac{1}{4}} L^{-\frac{3}{2}} \end{bmatrix}$$

$|H_1| = -\frac{3}{16} k^{-\frac{7}{4}} L^{\frac{1}{2}}$

$|H_2| = \left(\frac{3}{64} k^{-\frac{6}{4}} L^{-1}\right) - \left(\frac{1}{64} k^{-\frac{6}{4}} L^{-1}\right)$

$= \frac{2}{64}\left(k^{-\frac{6}{4}} L^{-1}\right)$

So it is negative definite because $|H_1| < 0$, so it is concave and, maximizing output

$|H_2| > 0$

Try any (k, L) $|H_1|$ is always negative and $|H_2|$ is always positive

global max - Local max