FN211 Optimization

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Outline

- Unconstrained Optimization
- Constrained Optimization

Economics and finance are filled with optimization problems

Consumers make buying decisions to maximize utility

Businesses make hiring/capital investment decisions to minimize costs

Businesses make pricing decisions to maximize profits

Consumers make labor decisions to maximize utility

Consumers make savings decisions to maximize utility

We need some tools to analyze these optimization problems

Example

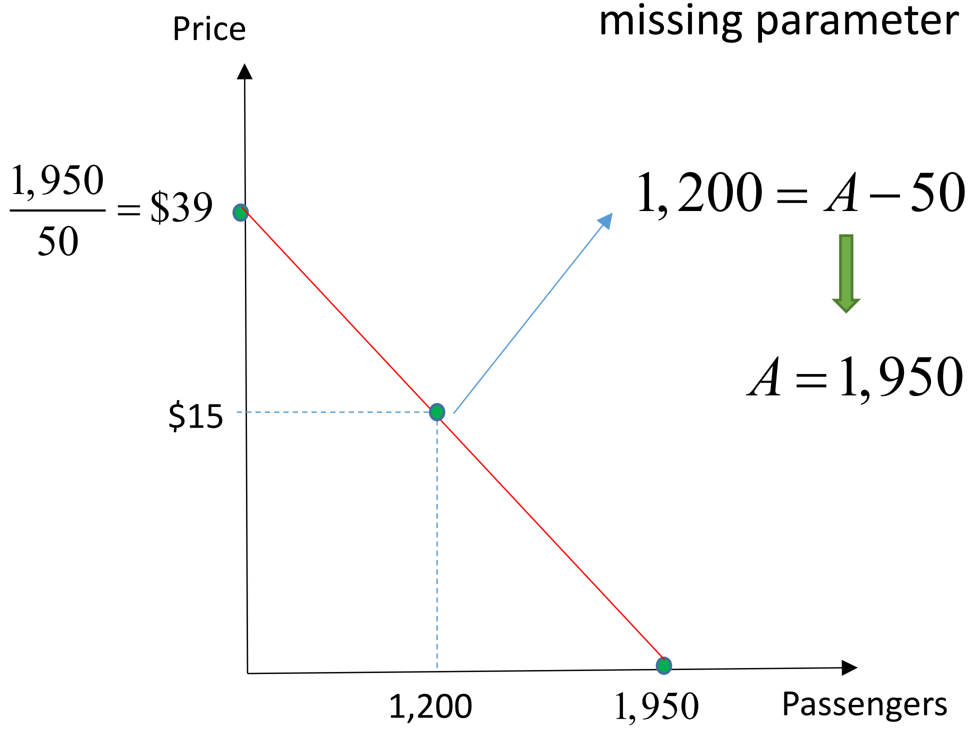
An airport shuttle currently charges $15 and carries an average of 1,200 passengers per day. It estimates that for each dollar it raises its fare, it loses an average of 50 passengers.

Current Revenues:

Could we do better?

Unconstrained Optimization

First, lets figure out the demand curve this business faces. For simplicity, lets assume that the demand is linear.



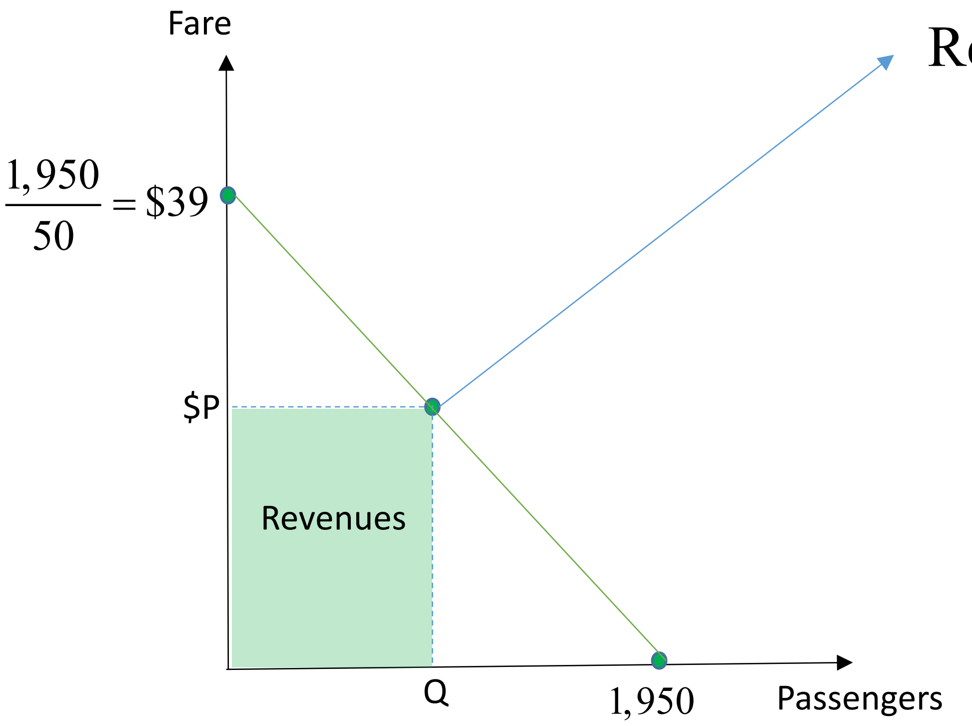
We can use the one data point we have to find the missing parameter

$$1,200 = A - 50(15)$$

$$A = 1,950$$

$$Q = 1,950 - 50P$$

The problem here is to maximize revenues.



$$\text{Revenues} = P * Q$$

$$\boxed{Q = 1,950 - 50P}$$

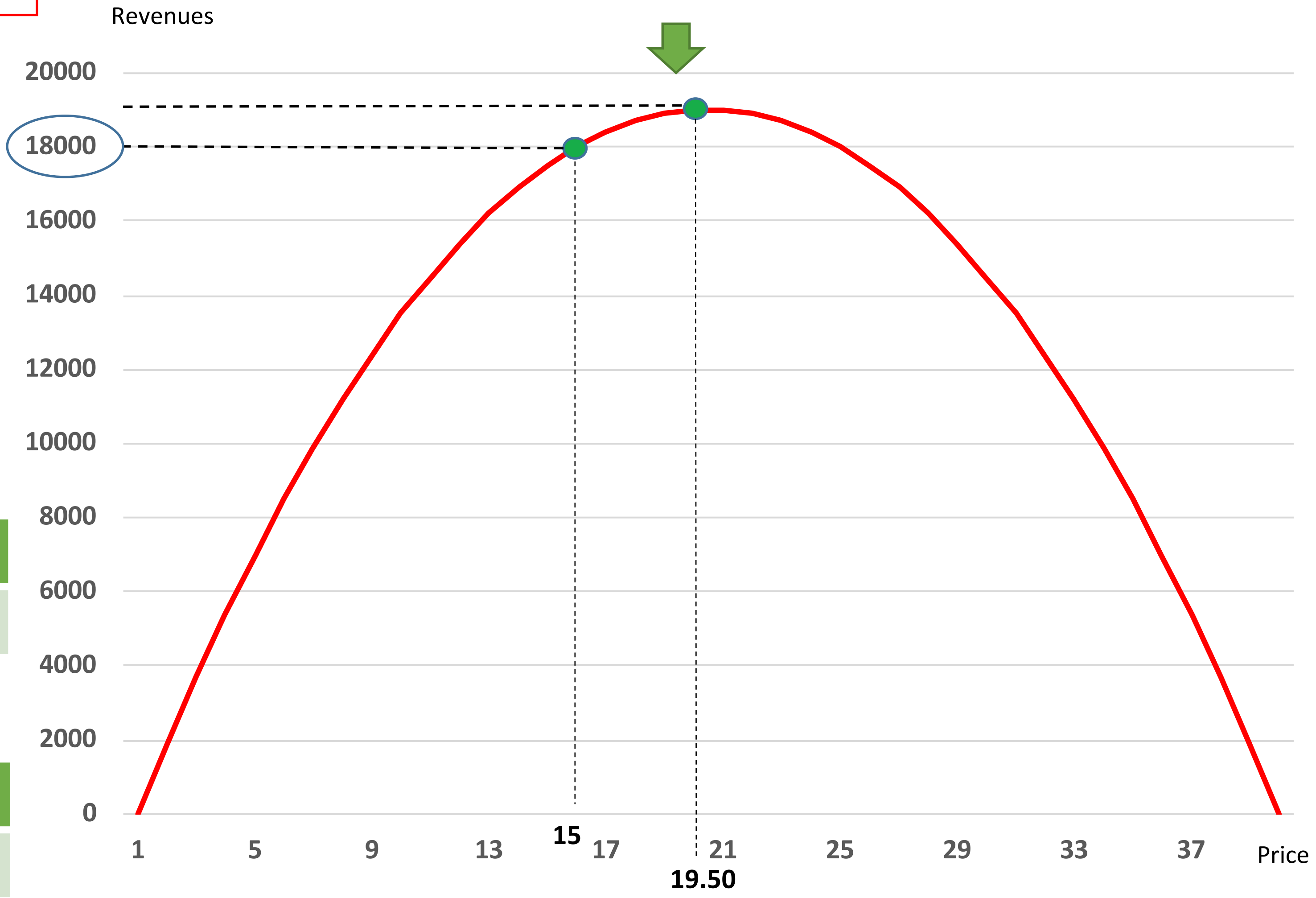
So, we have revenues as a function of the price charged. Now what?

$$\underbrace{Q = 1,950 - 50P}_{\text{Quantity}} \qquad \underbrace{1,950P - 50P^2}_{\text{Revenues}}$$

Price	Quantity	Revenues
0	1,950	0
1	1,900	1,900
2	1,850	3,700
3	1,800	5,400

Price	Quantity	Revenues
19.50	975	19,012.50

Price	Quantity	Revenues
36	150	5,400
37	100	3,700
38	50	1,900
39	0	0

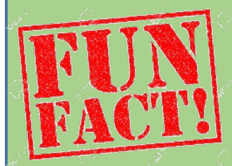


We could do better! Now, how do we find this point without resorting to excel?

Pierre de Fermat
1607- 1665

“Take the derivative and set it equal to zero!”

-Fermat’s Theorem

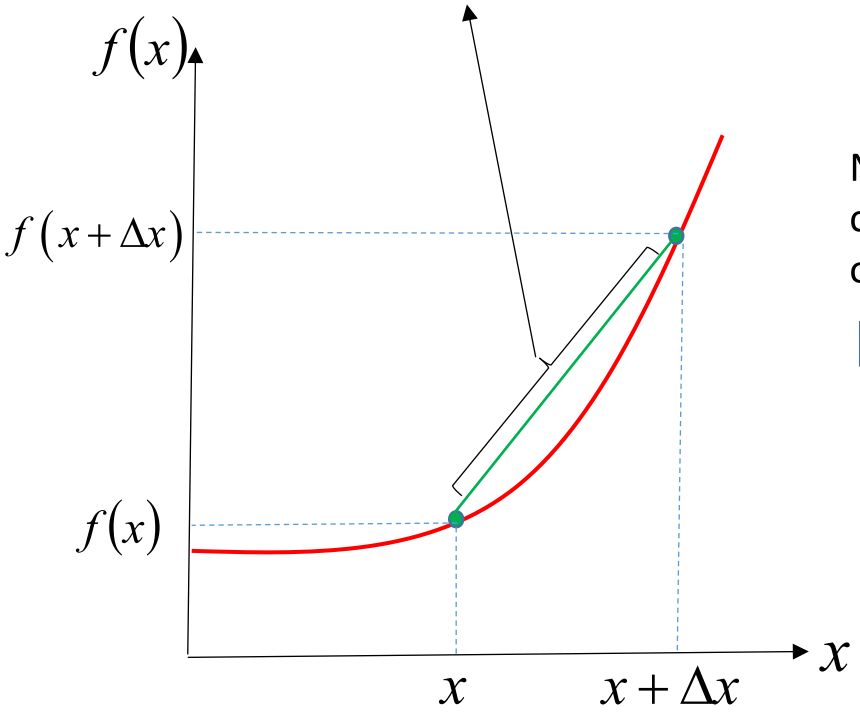


Pierre de Fermat was actually a lawyer before he was a mathematician….perhaps he was trying to figure out how to maximize his legal fees!

But, what’s a derivative and why set it equal to zero?

Imagine calculating the slope between two distinct points on a function

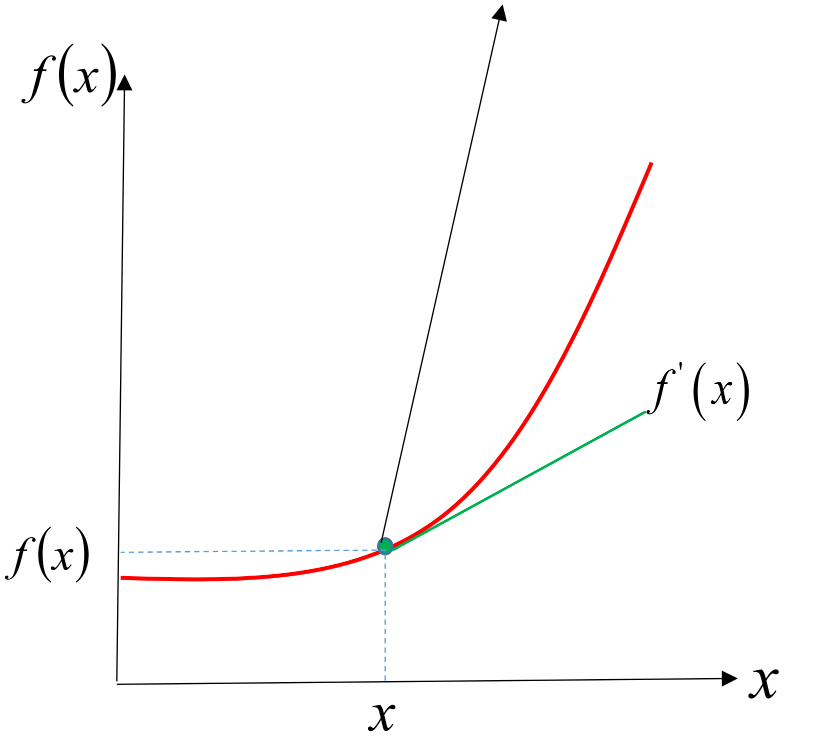
$$slope = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$



Now, let those two points get closer and closer to each other

The derivative is the result of the distance between those two points approaching zero

$$slope = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right] = f'(x)$$

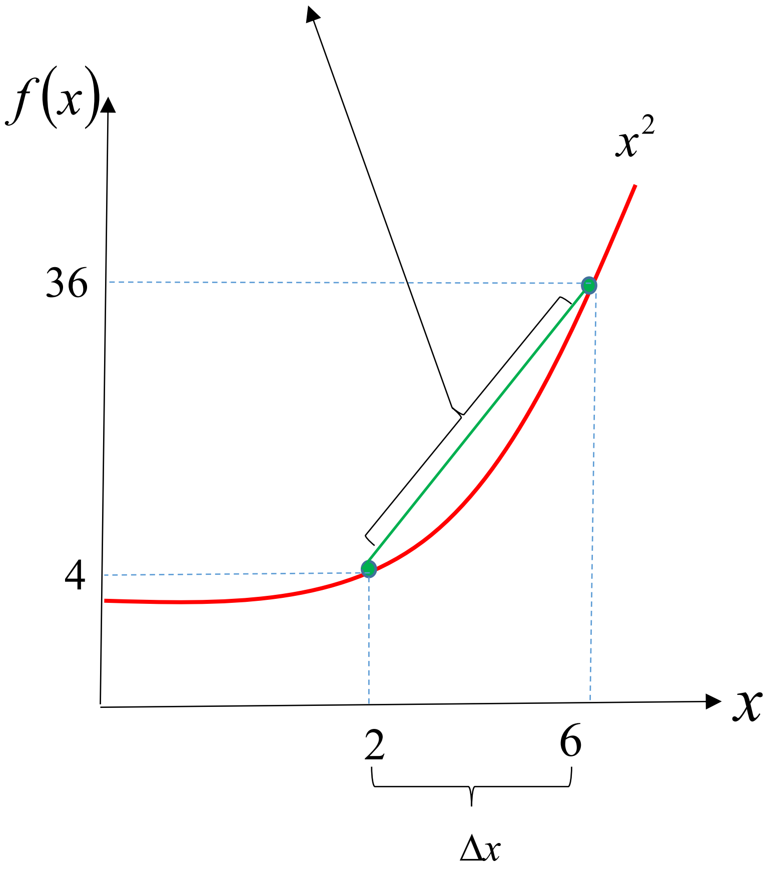


Let's try one numerically…

$$slope = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

2

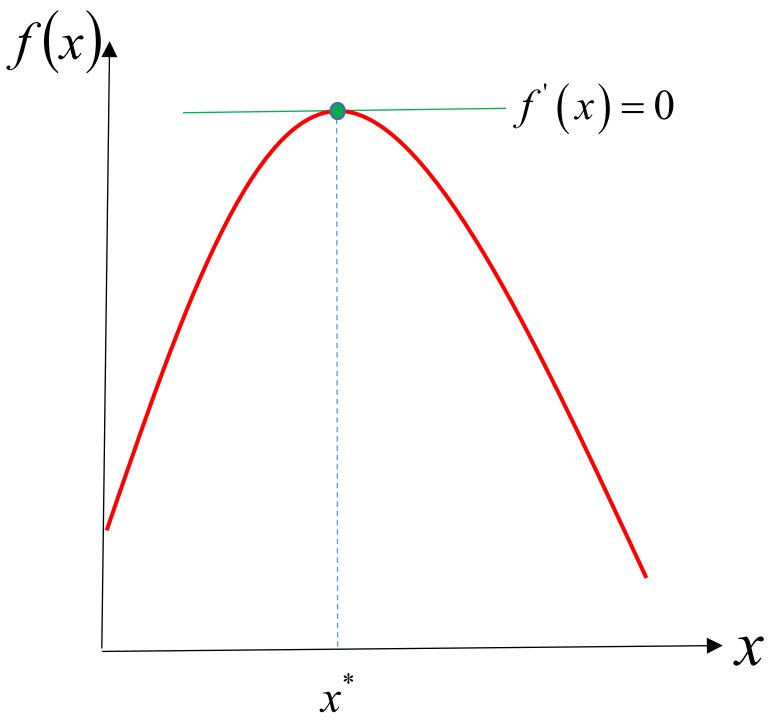
$$slope = \frac{36-4}{4} = 8$$



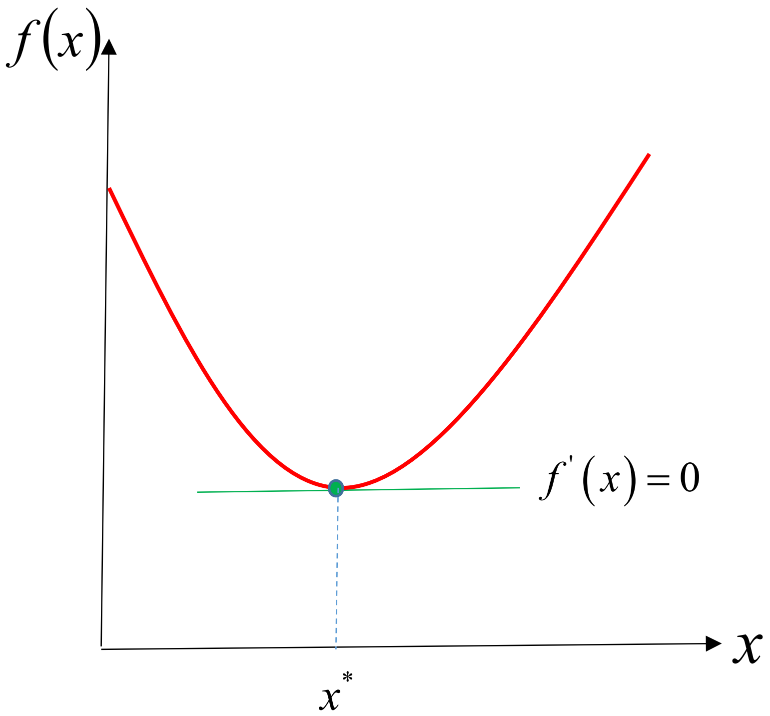
Δx	$f(x+\Delta x)$	**Slope**
4	36	8
2	16	6
1	9	5
.5	6.25	4.5
.25	5.0625	4.25
.1	4.41	4.1
.05	4.2025	4.05
.01	4.0401	4.01
.001	4.004001	4.001

$$slope = 2x$$
$$f'(2) = 4$$

A necessary condition for a maximum or a minimum is that the derivative equals zero

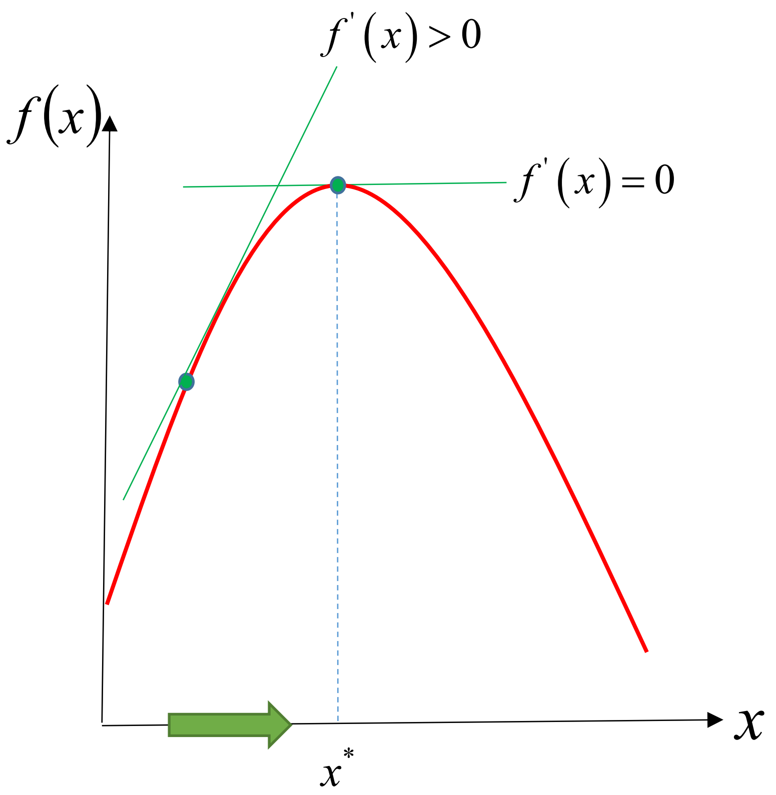


OR

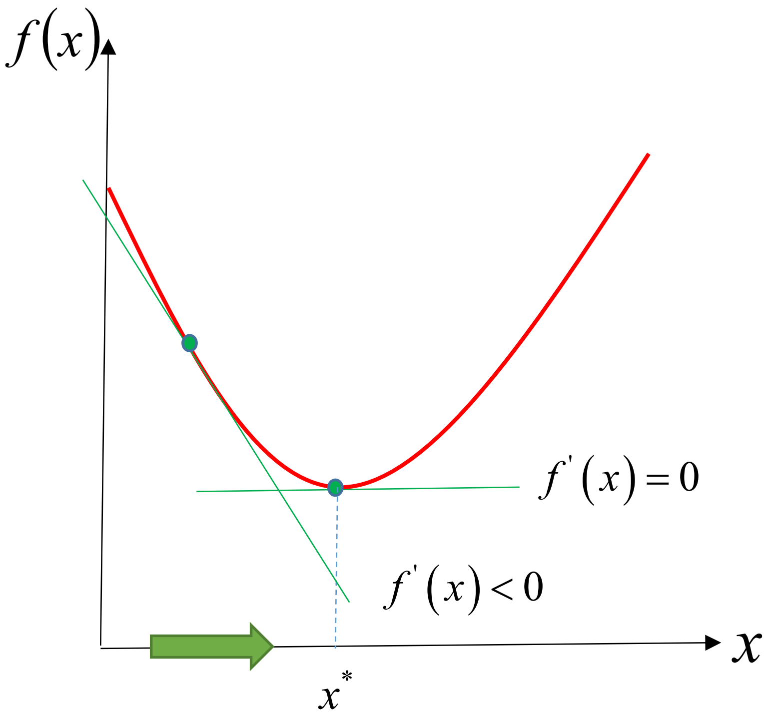


But how can we tell which is which?

While the first derivative measures the slope (change in the value of the function), the second derivative measures the change in the first derivative (change in the slope)

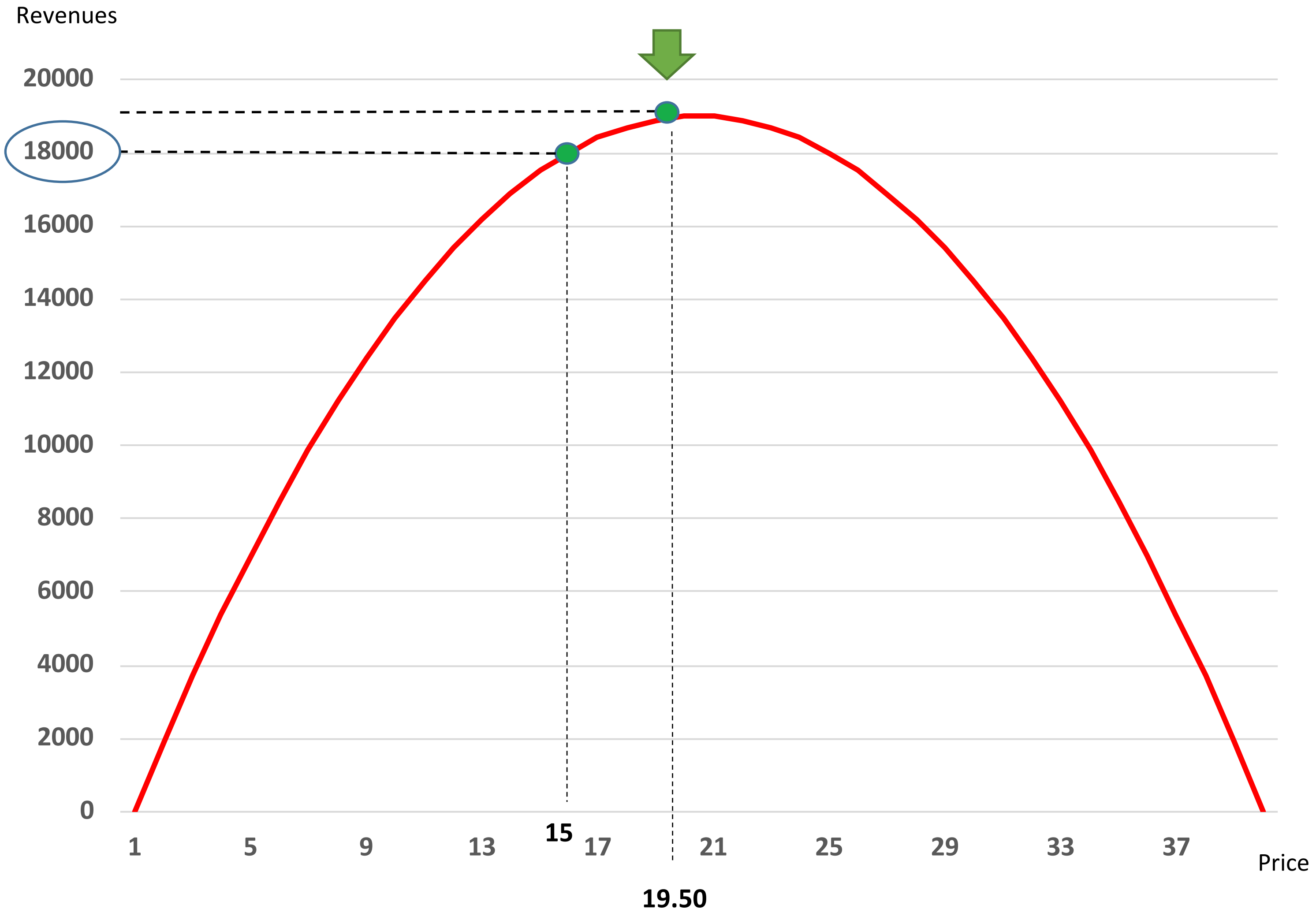


As x increases, the slope is decreasing



As x increases, the slope is increasing

Back to our revenue problem….



$$R = 1,950P - 50P^2$$

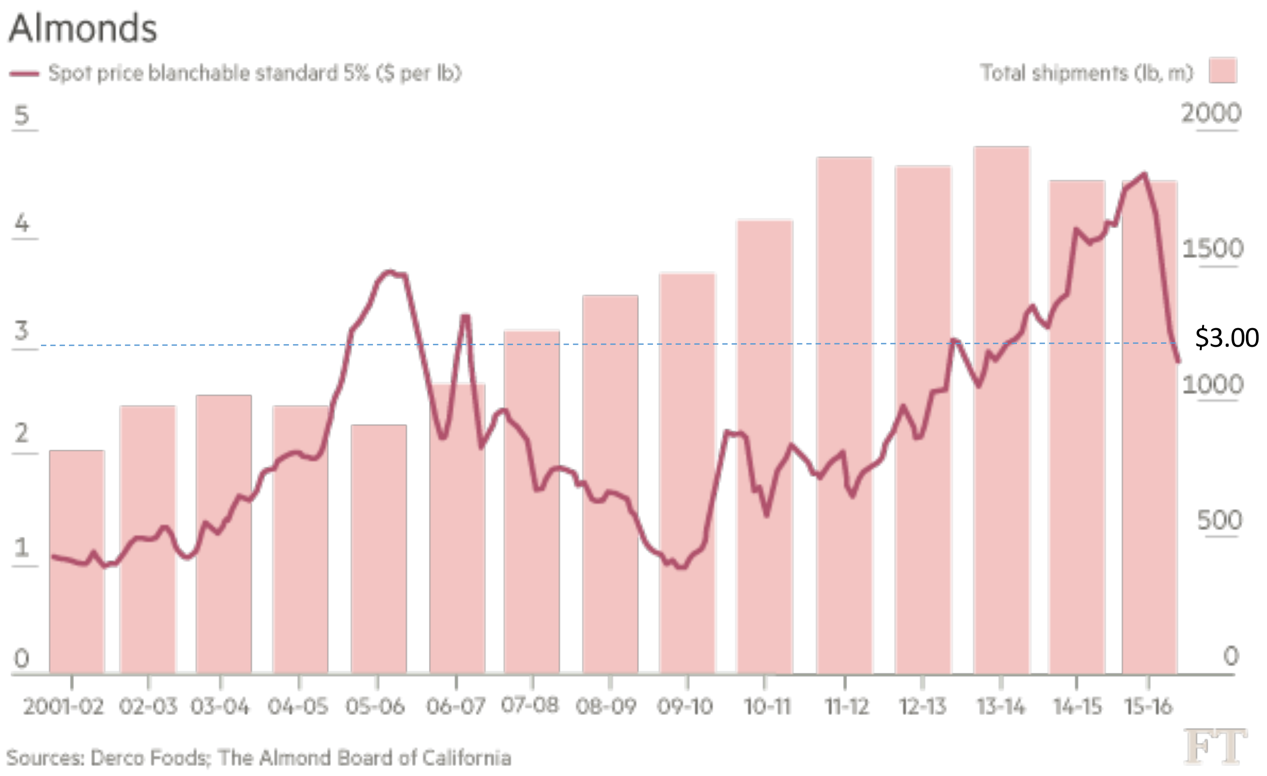
Take the derivative with respect to 'P'

Set the derivative equal to zero and solve for 'P'

$$1,950 - 100P = 0$$

Note, the second derivative is negative…a maximum!

After prices for almonds climbed to a record $4 per pound in 2014, farmers across California began replacing their cheaper crops with the nut, causing a huge increase in supply. Now, the bubble has popped. Since late 2014, according to The Washington Post, almond prices have fallen by around 25%.



Lets suppose that almond prices are currently $3/lb. and are falling at the rate of $.04/lb. per week.

Let's also suppose that your orchard is current bearing 35 pounds per tree, and that number will increase by 1 lb. per week.

How long should you wait to harvest your nuts to maximize your revenues?

We want to maximize revenues which is price per pound times total pounds sold

Now, take a derivative with respect to 't' and set it equal to zero

Then, solve for 't'

$$P = \$2.20$$
$$Q = 55$$
$$R = \$121$$

Suppose that you run a trucking company. You have the following expenses.

Expenses

- Driver Salary: \$22.50 per hour
- \$0.27 per mile depreciation
- v/140 dollars per mile for fuel costs where 'v' is speed in miles per hour

What speed should you tell your driver to maintain in order to minimize your cost per mile?

What speed should you tell your driver to maintain in order to minimize your cost per mile?

Note that if I divide the driver's salary by the speed, I get the driver's salary per mile

$$\frac{\frac{\$}{hr.}}{\frac{miles}{hr.}} = \frac{\$}{mile}$$

Now, take a derivative with respect to 't' and set it equal to zero

Then, solve for 't'

$$v = \sqrt{3,150} \cong 56\ mph$$

$$Cost \cong \$1.07\ per\ mile$$

Suppose you know that demand for your product depends on the price that you set **and** the level of advertising expenditures.

$$Q(p,A)=5{,}000-10p+40A+pA-.8A^2-.5p^2$$

Choose the level of advertising AND price top maximize sales

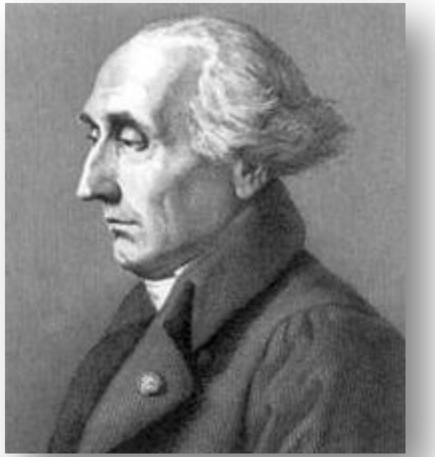
Choose the level of advertising AND price to maximize sales

$$Q = 5,000 - 10p + 40A + pA - .8A^2 - .5p^2$$

Now, we need partial derivatives with respect to both ‘p’ and ‘A’. Both are set equal to zero

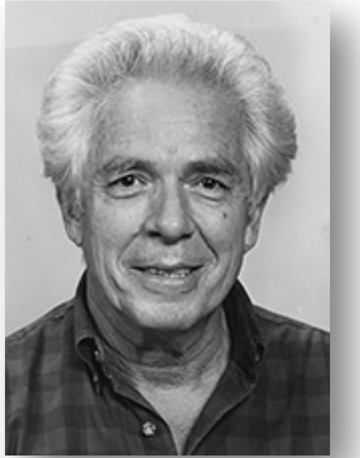
This gives us two equations with two unknowns

Constrained Optimization

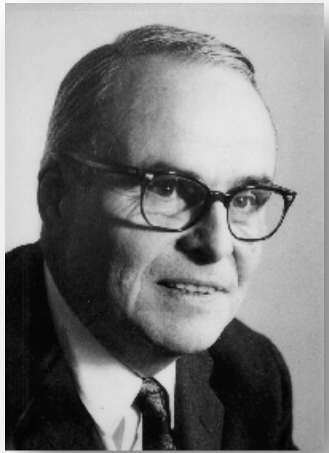


Joseph-Louis Lagrange
1736-1813

The method of Lagrange multipliers is a strategy for finding the local maxima and minima of a function subject to equality constraints.

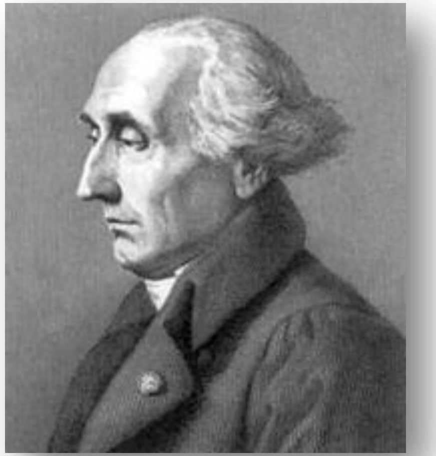


Harold W. Kuhn
1925 - 2014



Albert W. Tucker
1905 - 1995

Harold Kuhn and Albert Tucker later extended this method to inequality constraints



The Lagrange method writes the constrained optimization problem in the following form

Objective

Constraint(s)

Choice variables

$$\max_{x,y} f(x,y)$$

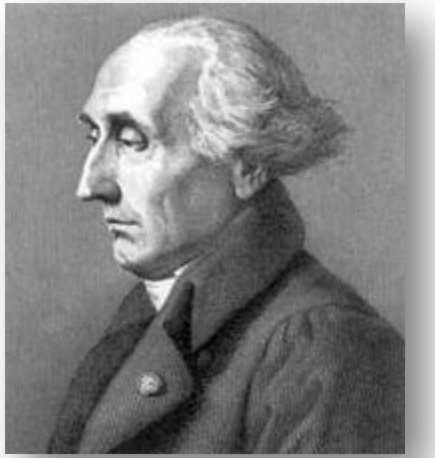
$$subject\ to \quad g(x,y) \geq 0$$

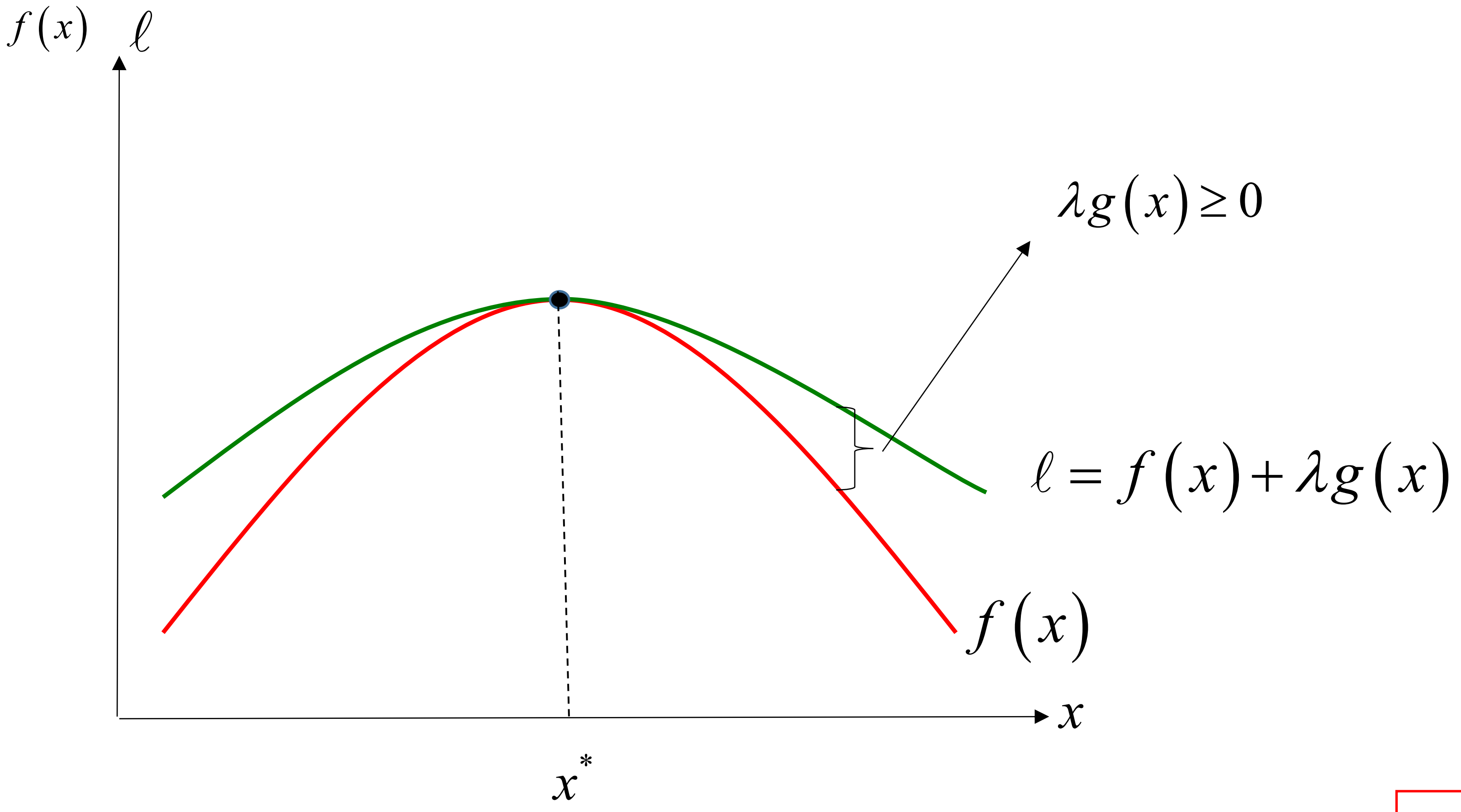
The problem is then rewritten as follows

$$\ell = f(x,y) + \lambda g(x,y)$$

Multiplier (assumed greater or equal to zero)

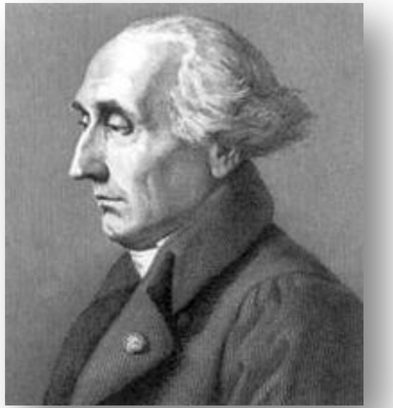
Here' s how this would look in two dimensions…we've created a new function that includes the constraints. This new function coincides with the original objective at one point – the maximum of both!





Therefore, at the maximum, two thigs have to be true

$\ell_x = 0$	The new function is maximized
$\lambda g(x) = 0$	The new function coincides with the original objective



So, we have our Lagrangian function….

$$\ell = f(x,y) + \lambda g(x,y)$$

We need the derivatives with respect o both ‘x’ and ‘y’ to be zero

$$\boxed{\begin{aligned} \ell_x &= f_x(x,y) + \lambda g_x(x,y) = 0 \\ \ell_y &= f_y(x,y) + \lambda g_y(x,y) = 0 \end{aligned}}$$

And then we have the “multiplier conditions”

$$\boxed{\lambda \geq 0 \qquad g(x,y) \geq 0 \qquad \lambda g(x,y) = 0}$$

Example: Suppose you sell two products (X and Y). Your profits as a function of sales of X and Y are as follows:

$$\text{Profit} = 10x + 20y - .1(x^2 + y^2)$$

Objective

$f(x, y)$

Your production capacity is equal to 100 total units. Choose X and Y to maximize profits subject to your capacity constraints.

$$x + y \leq 100$$

Constraint

$g(x, y)$

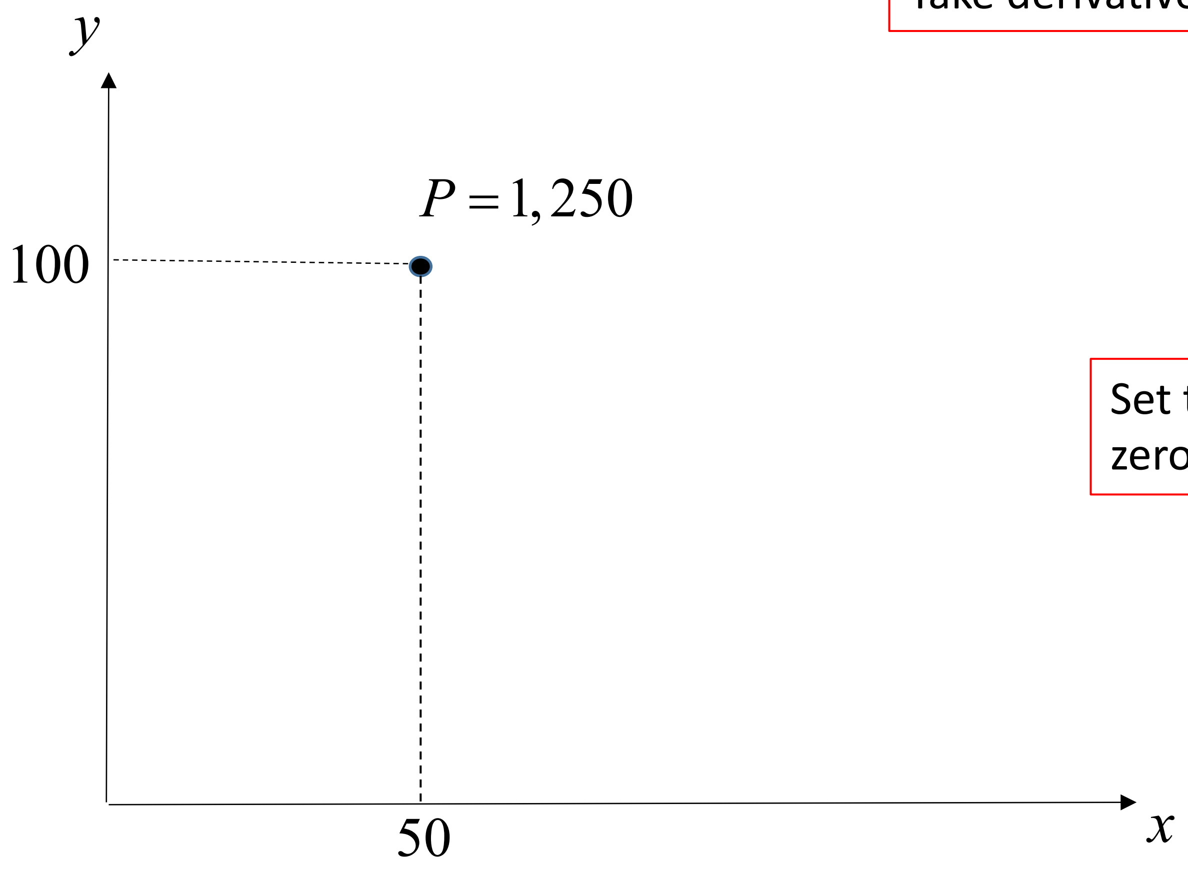
First, for comparison purposes, lets solve this unconstrained….

$$\text{Profit} = 10x + 20y - .1(x^2 + y^2)$$

Take derivatives with respect to 'x' and 'y'

Set the derivatives equal to zero and solve for 'x' and 'y'

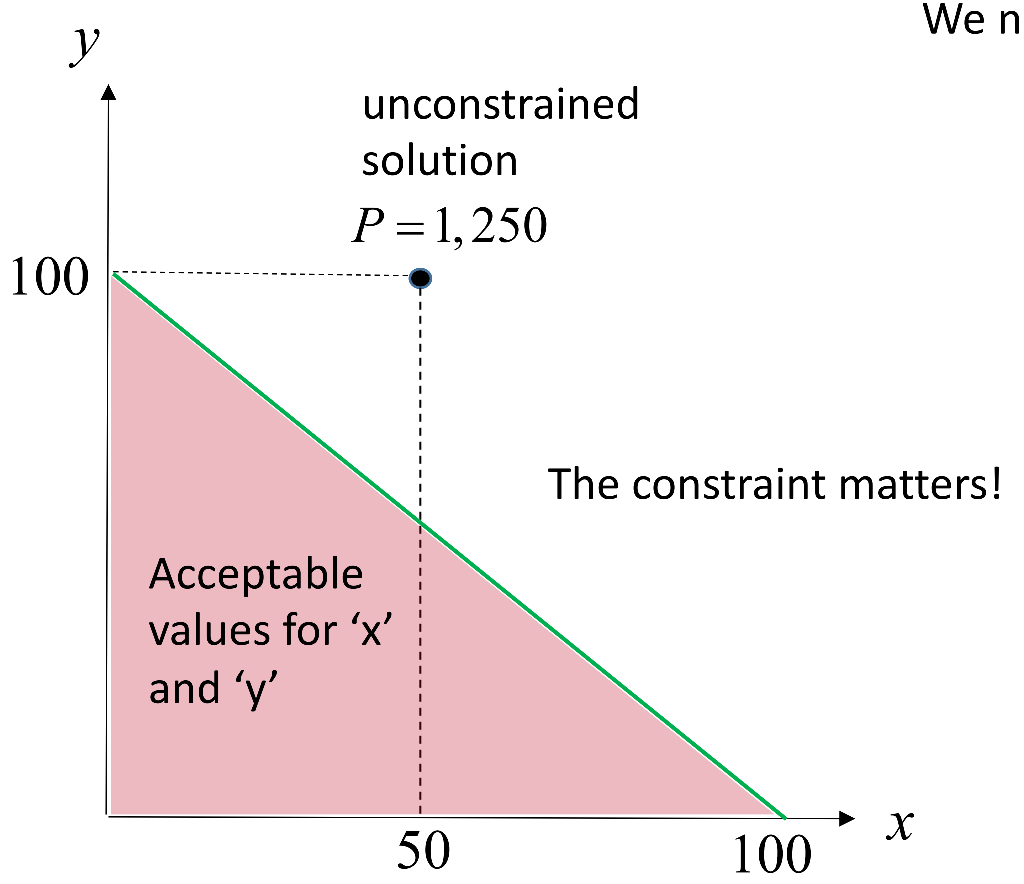
$$x^* = 50$$
$$y^* = 100$$
$$P = 1,250$$



Now, add the constraint

$$\text{Profit} = 10x + 20y - .1(x^2 + y^2)$$

Subject to $$x + y \leq 100$$



We need to get the problem in the right format….we need

$$g(x, y) \geq 0 \Rightarrow x + y \leq 100$$

Subtract 'x' and 'y' from both sides

Now, we can write the lagrangian

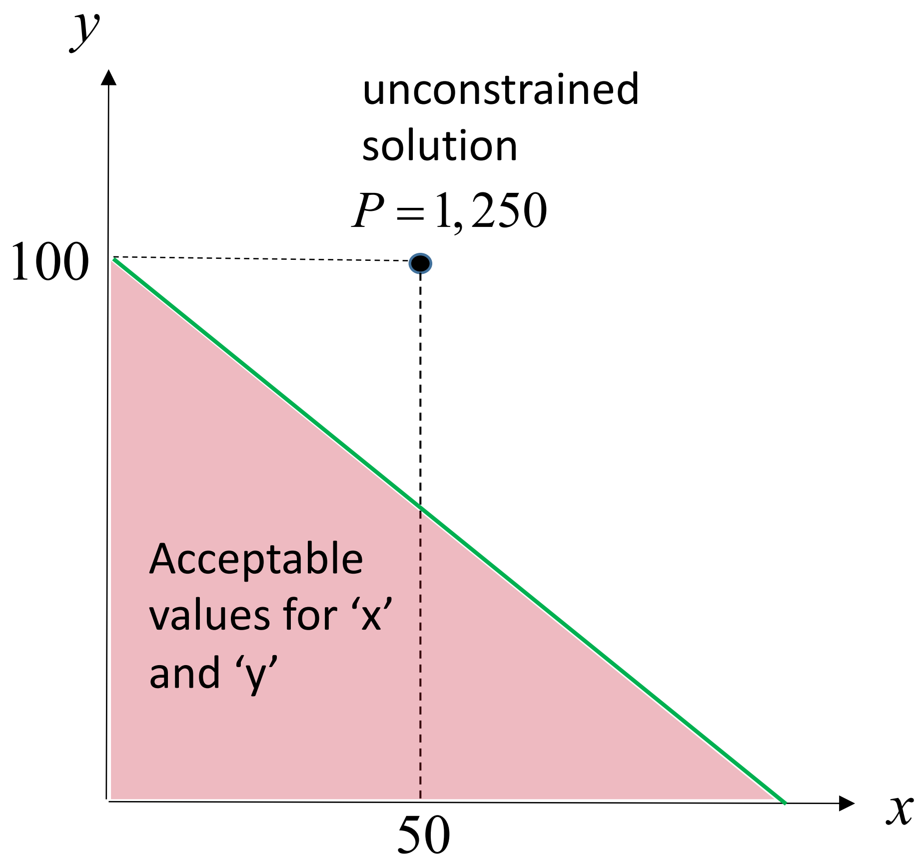
$$\ell = \underbrace{\boxed{10x + 20y - .1(x^2 + y^2)}}_{\text{Objective}} + \underbrace{\lambda}_{\text{Multiplier}} \underbrace{\boxed{(100 - x - y)}}_{\text{Constraint}}$$

Now, the mechanical part

$$\ell = 10x + 20y - .1(x^2 + y^2) + \lambda(100 - x - y)$$

Take derivatives with respect to ‘x’ and ‘y’

And the multiplier conditions

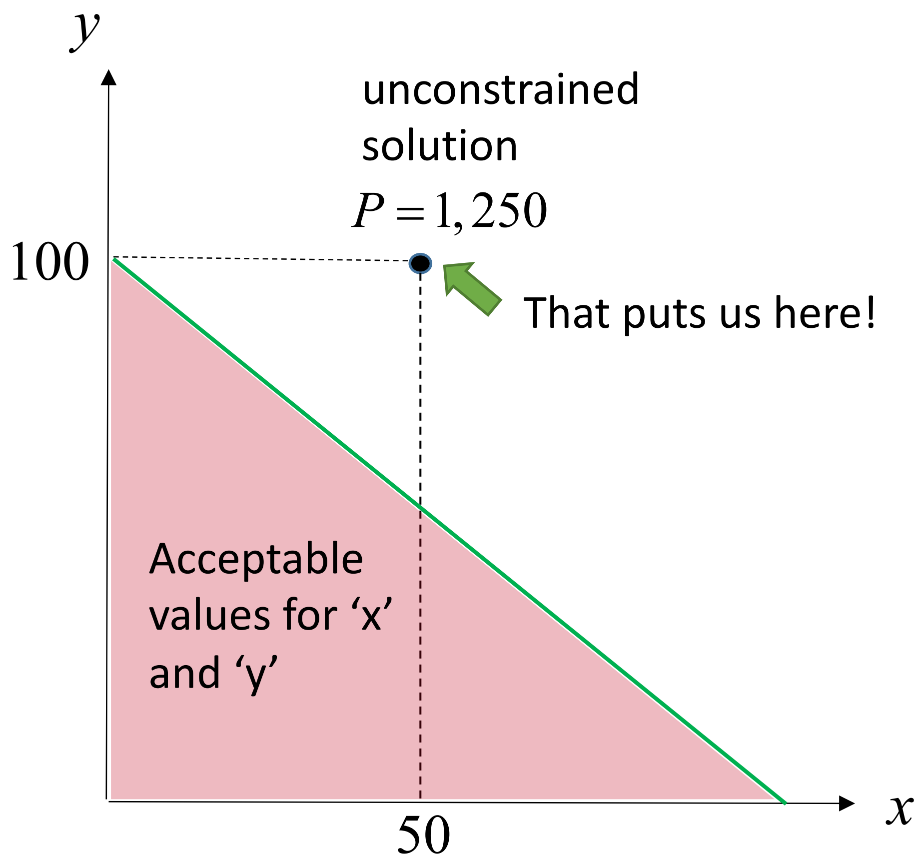


Now, the mechanical part

$$\ell_x = 10 - .2x - \lambda = 0$$

$$\ell_y = 20 - .2y - \lambda = 0$$

$$\lambda \geq 0 \qquad 100 - x - y \geq 0 \qquad \lambda(100 - x - y) = 0$$



First, let's suppose that lambda equals zero

$$\left.\begin{array}{l} 10 - .2x = 0 \\ 20 - .2y = 0 \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} x^* = 50 \\ y^* = 100 \end{array}}$$

Now, the mechanical part

$$\ell_x = 10 - .2x - \lambda = 0 \qquad \lambda > 0 \qquad \lambda(100 - x - y) = 0 \Rightarrow 100 - x - y = 0$$

$$\ell_y = 20 - .2y - \lambda = 0$$

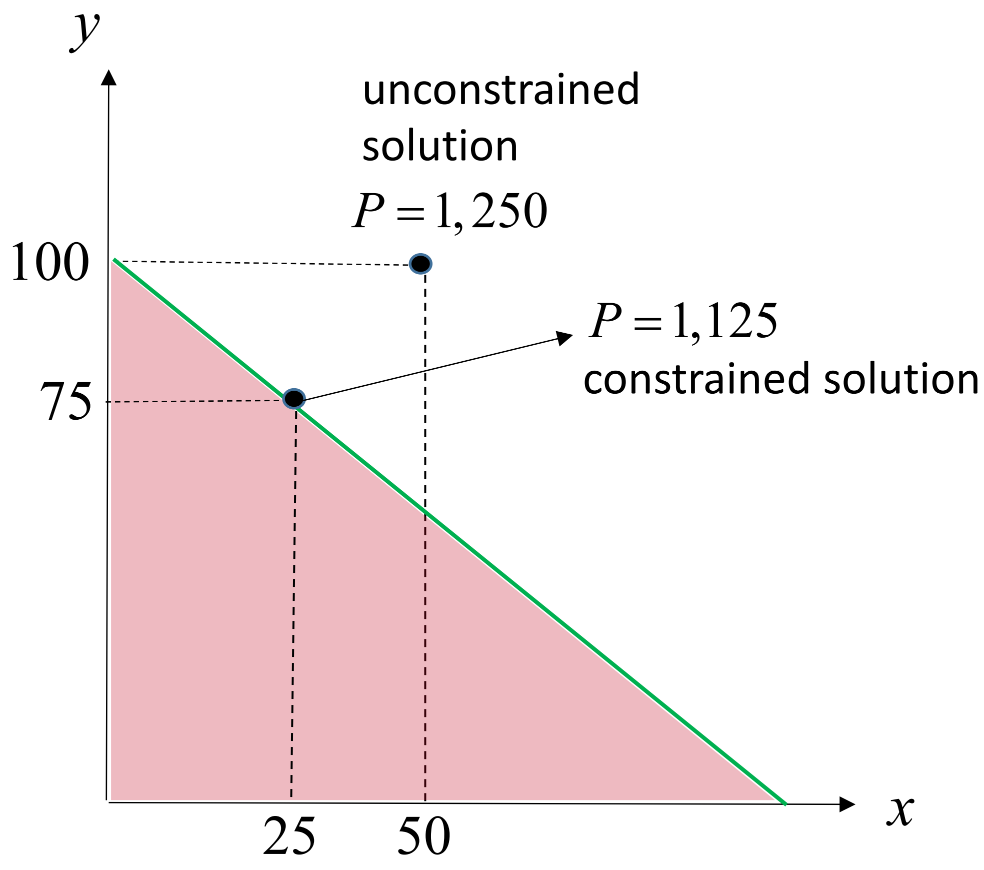
So, we have three equations and three unknowns (‘x’, ‘y’, and lambda)

$$\left.\begin{aligned} 10 - .2x - \lambda &= 0 \\ 20 - .2y - \lambda &= 0 \end{aligned}\right\} \Rightarrow$$

Solve the first two expressions for lambda

rearrange

Plug into third equation

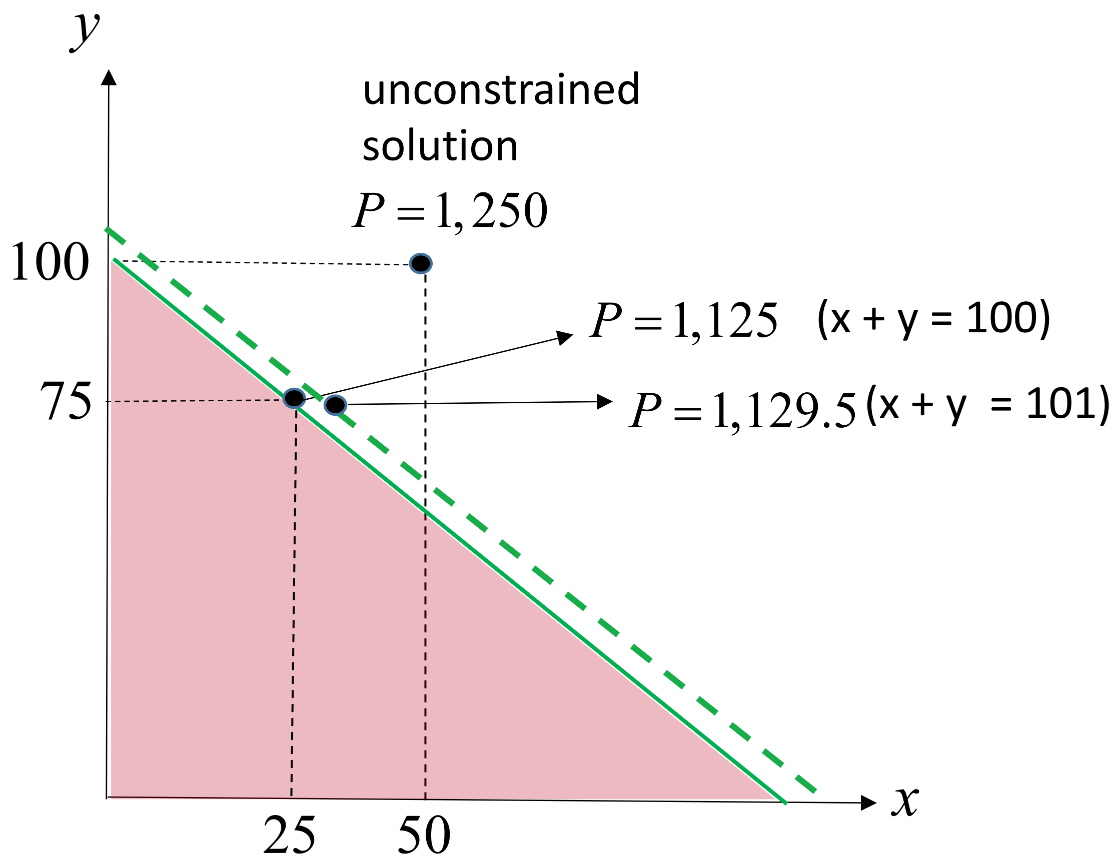


Suppose that the production constraint increases from 100 to 101

$$\ell_x = 10 - .2x - \lambda = 0$$

$$\ell_y = 20 - .2y - \lambda = 0$$

$$\lambda > 0 \qquad \lambda(101 - x - y) = 0 \Rightarrow 101 - x - y = 0$$

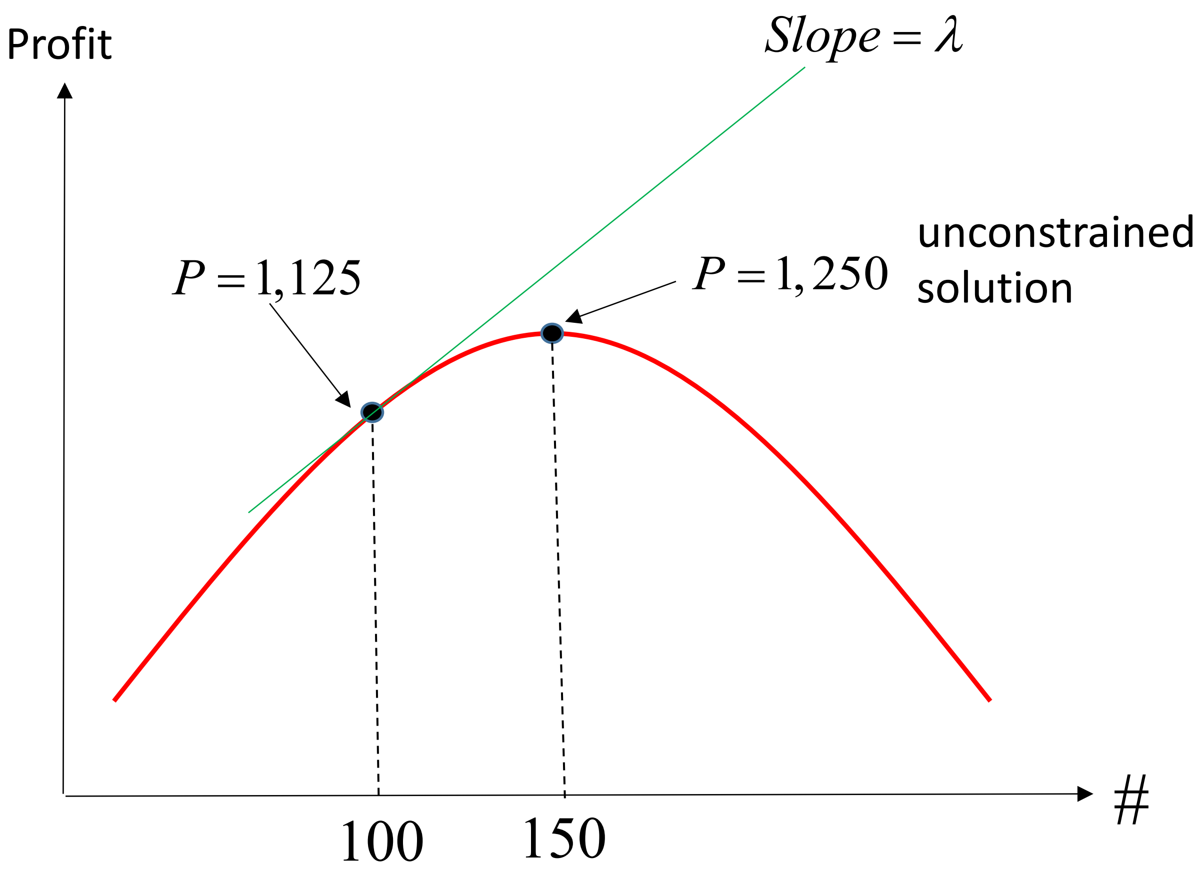


Resolve….

$$101 - x - (x + 50) = 0 \Rightarrow$$

We optimize once again and, given the extra capacity, profits up by ……. This is pretty close to the value of lambda!

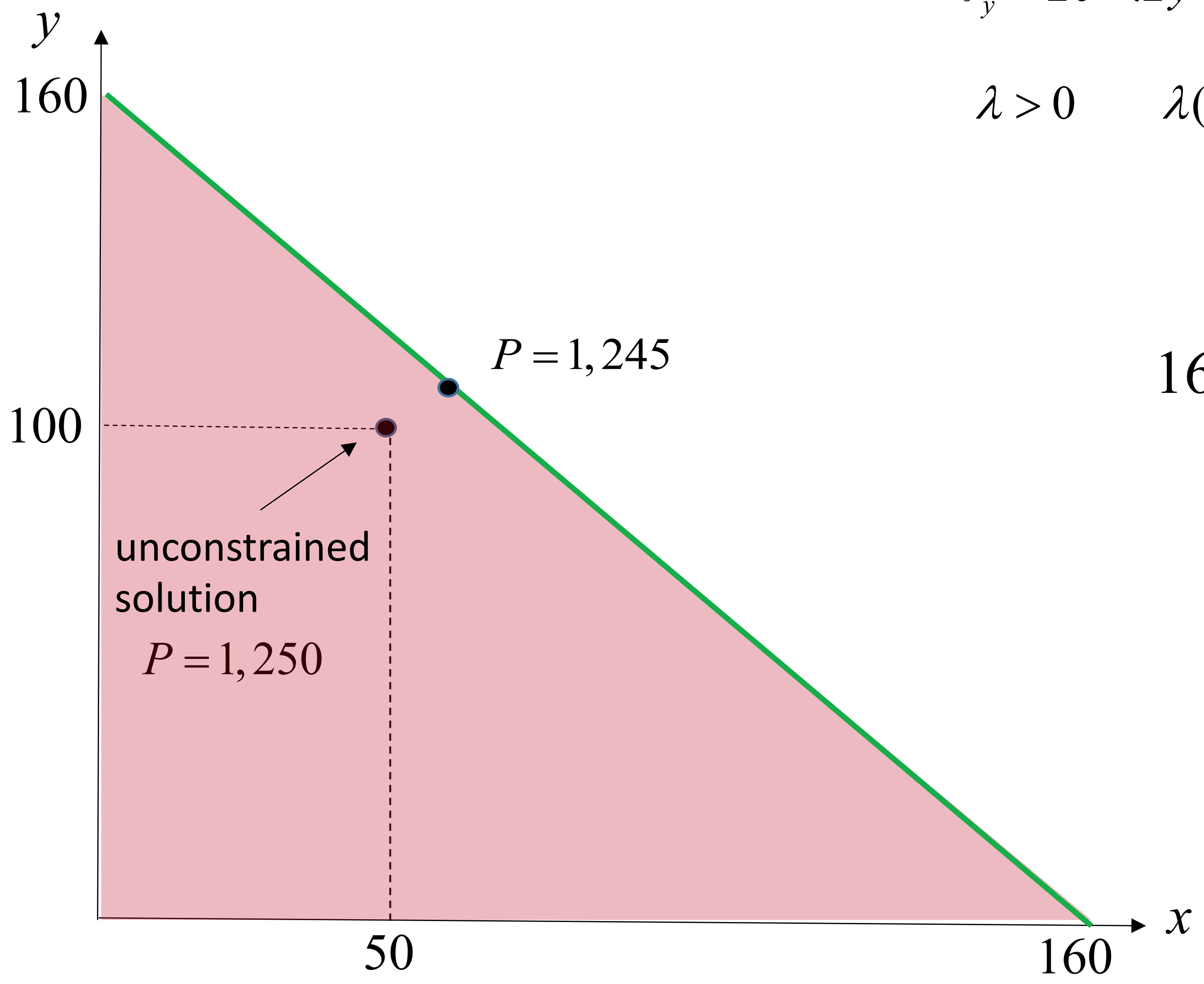
Let's plot profits (optimized subject to the constraint) as a function of the constraint



$$x + y \leq \#$$

Lambda measures the marginal impact of the constraint on the objective function!

Suppose that the production constraint is 160



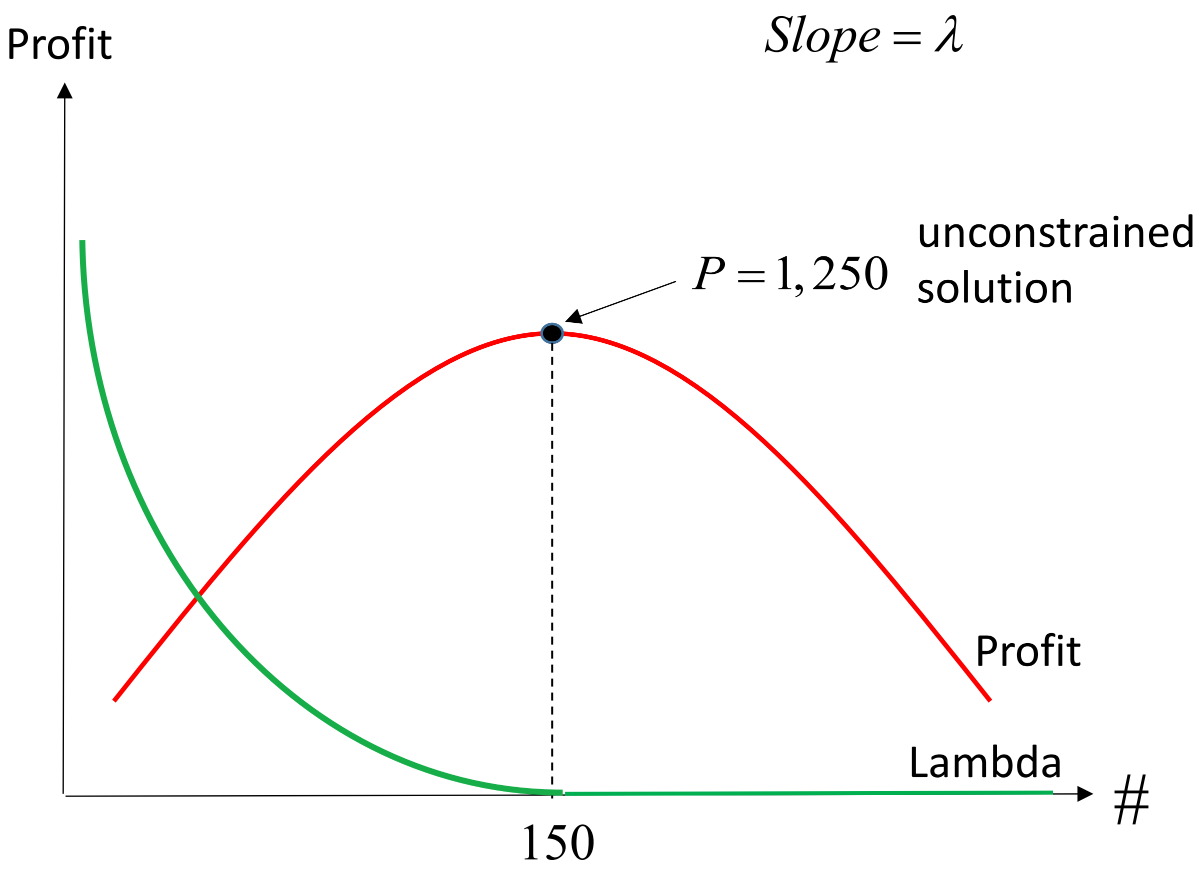
If we continue to assume lambda is positive….

$$\ell_x = 10 - .2x - \lambda = 0$$
$$\ell_y = 20 - .2y - \lambda = 0$$

$$\lambda > 0 \qquad \lambda(160 - x - y) = 0 \Rightarrow 160 - x - y = 0$$

$$160 - x - (x + 50) = 0 \Rightarrow$$

Lambda measures the marginal impact of the constraint on the objective function! So, when the constraint in irrelevant, lambda hits zero!



$$\boxed{x + y \le \#}$$

Example

Postal regulations require that a package whose length plus girth exceeds 108 inches must be mailed at an oversize rate. What size package will maximize the volume while staying within the 108 inch limit?

$$Girth = 2x + 2y$$

$$Volume = xyz$$

$$\max_{x>0,y>0,z>0} \{xyz\}$$

$$subject\ to \qquad 2x + 2y + z \leq 108$$

$$\boxed{g(x,y,z) \geq 0 \implies 108 - 2x - 2y - z \geq 0}$$

Remember, we need to write the constraint in the right format!

Postal regulations require that a package whose length plus girth exceeds 108 inches must be mailed at an oversize rate. What size package will maximize the volume while staying within the 108 inch limit?

First, write down the lagrangian

$$\ell = xyz + \lambda(108 - 2x - 2y - z)$$

Now, take the derivatives with respect to ‘x’, ‘y’, and ‘z’

Set the derivatives equal to zero

Multiplier conditions

Lets assume that lambda is positive

That leaves us four equations and 4 unknowns

Adding 1 inch to the girth/length will allow us to increase the area by approximately 324 cubic inches

Suppose that you are choosing purchases of apples and bananas. Your total satisfaction as a function of your consumption of apples and bananas can be written as

$$Utility = A^{.4}B^{.6}$$

Happiness

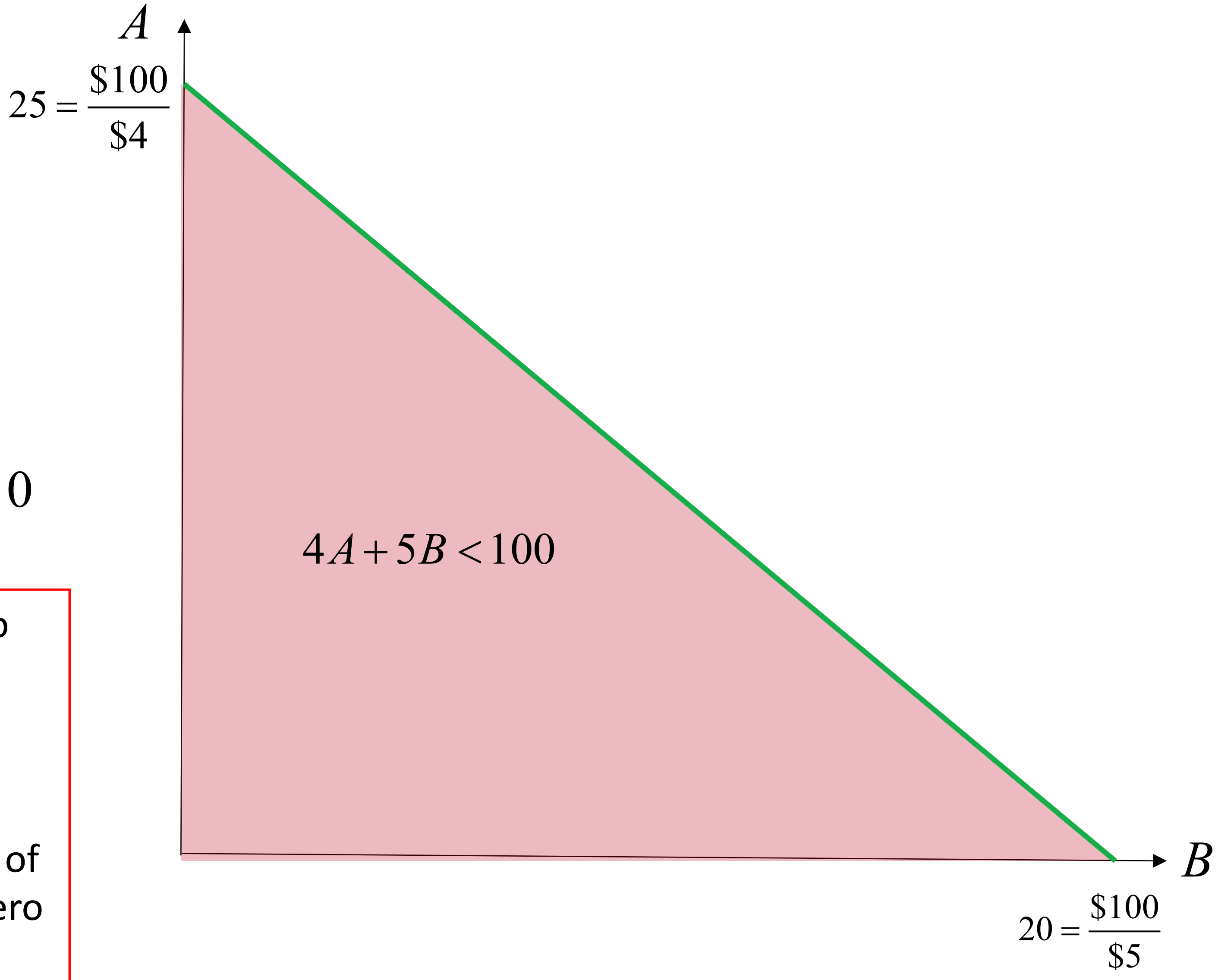
Weekly Apple Consumption

Weekly Banana Consumption

Apples cost \$4 each and bananas cost \$5 each. You want to maximize your satisfaction given that you have \$100 to spend

$$Total\ Expenditures = 4A + 5B$$

$$Total\ Expenditures = 4A + 5B = 100$$



$$\max_{A \geq 0, B \geq 0} A^{.4} B^{.6}$$

$$subject\ to \quad 100 - 4A - 5B \geq 0$$

*Note: Technically, there are two additional constraints here

$$A \geq 0 \quad B \geq 0$$

But given that zero consumption of either apples or oranges yields zero utility, we can ignore them

Apples cost \$4 each and bananas cost \$5 each. You want to maximize your satisfaction given that you have \$100 to spend

$$\max_{A\geq 0, B\geq 0} A^{.4}B^{.6}$$

$$subject\ to\quad 100-4A-5B\geq 0$$

So, we set up the lagrangian again

$$\ell(A,B)=A^{.4}B^{.6}+\lambda(100-4A-5B)$$

Take derivatives with respect to ‘A’ and ‘B’ and set the derivatives equal to zero

Let’s again assume lambda is positive!

$$100-4A-5B=0$$

$$\lambda>0$$

So we have three equations and three unknowns

$$.4A^{-.6}B^{.6} - 4\lambda = 0$$
$$.6A^{.4}B^{-.4} - 5\lambda = 0$$
$$4A + 5B = 100$$

Lets solve the first two for lambda

$$.4A^{-.6}B^{.6} - 4\lambda = 0 \qquad .6A^{.4}B^{-.4} - 5\lambda = 0$$
$$\lambda = .1A^{-.6}B^{.6} \qquad \lambda = .12A^{.4}B^{-.4}$$

Set the two expressions for lambda equal to each other and simplify

$$.1A^{-.6}B^{.6} = .12A^{.4}B^{-.4}$$
$$\boxed{B = 1.2A}$$

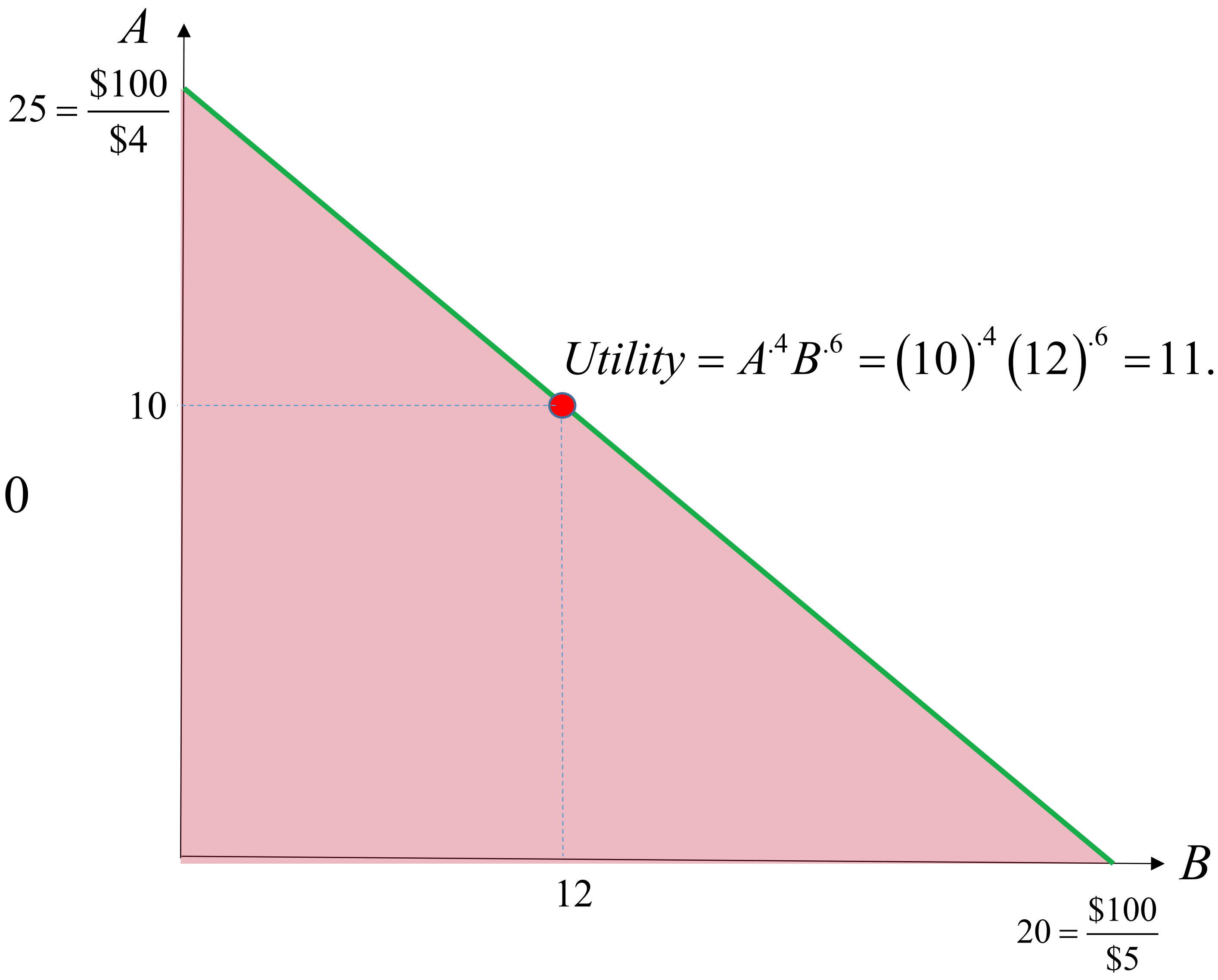
Plug into the last expression and simplify

$$4A + 5B = 100$$
$$4A + 5(1.2A) = 100$$
$$\boxed{\begin{aligned} A &= 10 \\ B &= 12 \end{aligned}}$$

$$Utility = A^{.4}B^{.6} = (10)^{.4}(12)^{.6} = 11.15$$
$$\lambda = .1A^{-.6}B^{.6} = .1(10)^{-.6}(12)^{.6} = .11$$

1 Additional dollar would provide an additional .11 units of happiness

$$Total\ \ Expenditures = 4A + 5B = 100$$



$$\max_{A>0,B>0} A^{.4}B^{.6}$$

$$subject\ to\ \ 100 - 4A - 5B \geq 0$$

Question?