Second-Order Conditions (for the case of multiple-independent variables).
(Unconstrained optimization).

Given $z = f(x, y)$.

F.O.C. $dz = 0 \iff f_x = f_y = 0$ where $f_x = \frac{\partial z}{\partial x}$; $f_y = \frac{\partial z}{\partial y}$.

S.O.C.: $d^2z < 0$ for maximum of z.
$d^2z > 0$ for minimum of z.

Note:
$$d^2z = f_{xx}dx^2 + f_{xy}\,dy\,dx + f_{xy}\,dx\,dy + f_{yy}\cdot dy^2 = [dx \;\; dy]\underbrace{\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{bmatrix}}_{H}\begin{bmatrix} dx \\ dy \end{bmatrix}$$

where $f_{xx} = \frac{\partial^2 z}{\partial x^2}$; $f_{xy} = f_{yx} = \frac{\partial^2 z}{\partial x \partial y}$; $f_{yy} = \frac{\partial^2 z}{\partial y^2}$.

The restrictions for the signs of f_{xy}, f_{xx}, and f_{yy}:

Given $d^2z = f_{xx}dx^2 + 2f_{xy}\,dx\,dy + f_{yy}dy^2$.

Define $q \equiv d^2z$, $u \equiv dx$, $v \equiv dx$, $a \equiv f_{xx}$, $b \equiv f_{yy}$, $h \equiv f_{xy}$

$$\Rightarrow q = au^2 + 2huv + bv^2$$
$$= a\left(u^2 + \frac{2h}{a}uv\right) + bv^2$$
$$= a\left(u^2 + \frac{2h}{a}u\cdot v + \frac{h^2v^2}{a^2}\right) - \frac{h^2v^2}{a} + bv^2$$
$$\therefore q = a\left(u + \frac{h}{a}v\right)^2 + \left(\frac{ab-h^2}{a}\right)v^2$$

Quadratic form:
$(u+\alpha v)^2 = u^2 + 2\alpha uv + \alpha^2 v^2$
here, $2\alpha = \frac{2h}{a}$
$\Rightarrow \alpha = \frac{h}{a}$
$\Rightarrow \alpha^2 = \frac{h^2}{a^2}$

$\Rightarrow q > 0$ iff $a > 0$ and $ab - h^2 > 0$
$q < 0$ iff $a < 0$ and $ab - h^2 > 0$.

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Since $q = d^2z$, $a = f_{xx}$, $b = f_{yy}$ and $h = f_{xy}$, we have:

1) $d^2z > 0$ <u>iff</u> $f_{xx} > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$; and

2) $d^2z < 0$ <u>iff</u> $f_{xx} < 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$.

<u>Notes:</u>

1) Since $f_{xx}f_{yy} - (f_{xy})^2 > 0 \Rightarrow f_{xx}f_{yy} > (f_{xy})^2 > 0.$,

sign of f_{yy} = sign of f_{xx}.

2) Alternatively we can write: $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$,

where $|H_1| = f_{xx}$ and $|H_2| = |H| = f_{xx}f_{yy} - (f_{xy})^2$.

Thus, the SOC's are:

For <u>max</u>, $d^2z < 0 \Leftrightarrow |H_1| < 0$ and $|H| > 0$;

For <u>min</u>, $d^2z > 0 \Leftrightarrow |H_1| > 0$ and $|H| > 0$.