Quiz 1: Date: April 19, 2022 from 11.00-12.30

Question 1 (10 Points)

Score.....................

Consider the one-period model of consumption and portfolio choice. Let an individual in this economy has the utility function as follow:

$$U(C) = ln(C)$$

Also, let $\frac{C_1}{C_0}$ is distributed as log-normal with mean equals μ_c and its variance is σ_c.
Please read and answer the following questions carefully and completely.

Score.....................

Question 1.1 (10 marks) Calculate the risk free rate R_f in terms of the individual's consumption, C_0 and C_1. Then,explain the relationship between the level of consumption and the risk free rate in this economy.

The individual solves the problem

$$\max_{C_0, \{w_i\}} U(C_0) + \delta E[U(C_1)]$$

$$\text{s.t. } C_1 = y_1 + (W_0 + y_0 - C_0)\sum_{i=1}^{n} w_i R_i$$

$$\sum_{i=1}^{n} w_i = 1$$

with a risk-free asset (asset f), then the F.O.C. can be written as

$$U'(C_0) = R_f \delta E[U'(C_1)]$$

$$\frac{1}{C_0} = R_f \delta E\left[\frac{1}{C_1}\right]$$

Q1.1) Con't

$$\frac{1}{R_f} = \delta E\left[\frac{C_0}{C_1}\right] = \delta E\left[\left(\frac{C_1}{C_0}\right)^{-1}\right]$$

$$R_f = \frac{1}{\delta} E\left[\frac{C_1}{C_0}\right] \quad \#$$

Intuitively, when the risk-free rate is high, so will the expected growth in consumption. $R_f \uparrow \Leftrightarrow E\left[\frac{C_1}{C_0}\right] \uparrow$

- The risk-free rate and expected growth in consumption are moving in the same direction. #

Score.....................

Question 1.2 (10 marks) Calculate the elasticity of intertemporal substitution in this setting. If in the next year, the interest rate is falling, Will the individual's consumption level increase or decrease? Why? To support your answer, use the concepts of income effect and substitution effect.

From Q1.1), we have $R_f = \frac{1}{\delta} E\left[\frac{C_1}{C_0}\right]$

Assume nonrandom labor income, and take log.

$$\ln(R_f) = -\ln\delta + \ln\left(\frac{C_1}{C_0}\right)$$

So, elasticity of intertemporal substitution is

$$\varepsilon = \frac{\partial \ln(C_1/C_0)}{\partial \ln(R_f)} = 1.$$

So, if the risk-free rate falls by 1% (percentage point),

the second period consumption will fall exactly 1% (percentage point).

This happens because the income effect and substitution effect <u>exactly offset</u> each other. #

Score.....................

Question 1.3 (10 marks) Solve for the pricing kernel P_i of any risky asset i in this economy. Then explain the meaning of this pricing kernel.

From the F.O.C. of the maximization problem, we have

$$P_i = E\left[\frac{\delta u'(c_1)}{u'(c_0)} X_i\right]$$

$$P_i = \delta E\left[\frac{c_0}{c_1} X_i\right] = \delta E\left[\left(\frac{c_1}{c_0}\right)^{-1} X_i\right] = \delta E\left[\frac{X_i}{c_1/c_0}\right]$$

So, the state where future consumption is high, the asset payoff (X_i) is not highly valued, resulting in relatively low price.

Nonetheless, in the state where future consumption is low, the asset payoff is highly valued, which leads to a relatively higher price. #

Score.....................

Question 1.4 (10 marks) Calculate Hansen-Jaganathan Bound and explain the meaning.

H-J Bound $\left|\frac{E[R_i]-R_f}{\sigma_{R_i}}\right| \leq \frac{\sigma_{m_{01}}}{E[m_{01}]}$

$$m_{01} = \frac{\delta U'(C_1)}{U'(C_0)} = \delta\left(\frac{C_1}{C_0}\right)^{-1} = \delta e^{\ln(C_1/C_0)^{-1}} = \delta e^{-\ln(C_1/C_0)}$$

$$\frac{\sigma_{m_{01}}}{E[m_{01}]} = \frac{\sqrt{E[e^{-2\ln(C_1/C_0)}] - E[e^{-\ln(C_1/C_0)}]^2}}{E[e^{-\ln(C_1/C_0)}]}$$

$$= \sqrt{E[e^{-2\ln(C_1/C_0)}] / E[e^{-\ln(C_1/C_0)}]^2 - 1}$$

By lognormal property of $\frac{C_1}{C_0}$.

$$= \sqrt{e^{-2\mu_c + \frac{1}{2}(-2)^2\sigma_c^2} / (e^{-\mu_c + \frac{1}{2}\sigma_c^2})^2 - 1}$$

$$= \sqrt{e^{-2\mu_c + 2\sigma_c^2} / e^{-2\mu_c + \sigma_c^2} - 1}$$

$$= \sqrt{e^{\sigma_c^2} - 1}$$

Hence, H-J Bound : $\left|\frac{E[R_i]-R_f}{\sigma_{R_i}}\right| \leq \sqrt{e^{\sigma_c^2} - 1}$ #

This implies that the Sharpe ratio is bounded by $\pm\sqrt{e^{\sigma_c^2}-1}$ where σ_c represents the volatility of the consumption.

That is $-\sqrt{e^{\sigma_c^2}-1} \leq$ Sharpe ratio $\leq \sqrt{e^{\sigma_c^2}-1}$ ← maximum Sharpe ratio possible.