





## CHAPTER 8


### Optimization without Constraint: More-Than-One Independent Variable Cases

 Conditions for maximum or minimum

 Third degree price discrimination → multi market

 Competitive Firm Input Choices: Cobb-Douglas Technology → multi factor of production

 Multiproduct-firm → competitive mkt  
→ monopoly

 Multiplant-firm

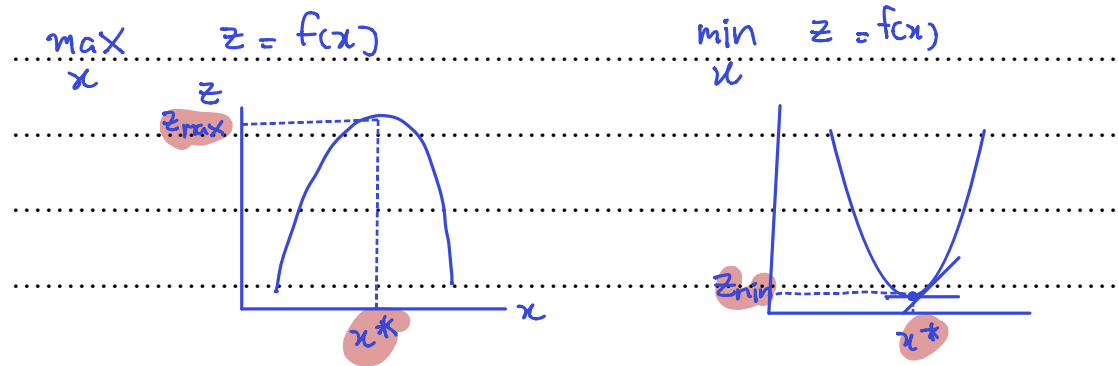


### Optimization Condition



### One choice variable

The objective function:  $z = f(x)$



Condition	Maximum	Minimum
First-order necessary (FONC, FOC)	Derivative $f'(x) = 0$ slope must be zero	Derivative $f'(x) = 0$
	Total differential $dz = f'(x)dx = 0$ $= 0 \quad > 0$	Total differential $dz = f'(x)dx = 0$
	$z$ doesn't change for extremely small increase (or decrease) in $x$	
Second-order sufficient (SOSC, SOC)	Derivative $f''(x) < 0$ slope is decreasing $(0 \rightarrow -)$	Derivative $f''(x) > 0$ slope is increasing $(0 \rightarrow +)$
$d^2z = d(dz)$ $= d(f'(x)dx)$ $= dx d(f'(x))$ $= dx(f''(x)dx)$ $= f''(x)(dx)^2$	Total differential $d^2z < 0$	Total differential $d^2z > 0$

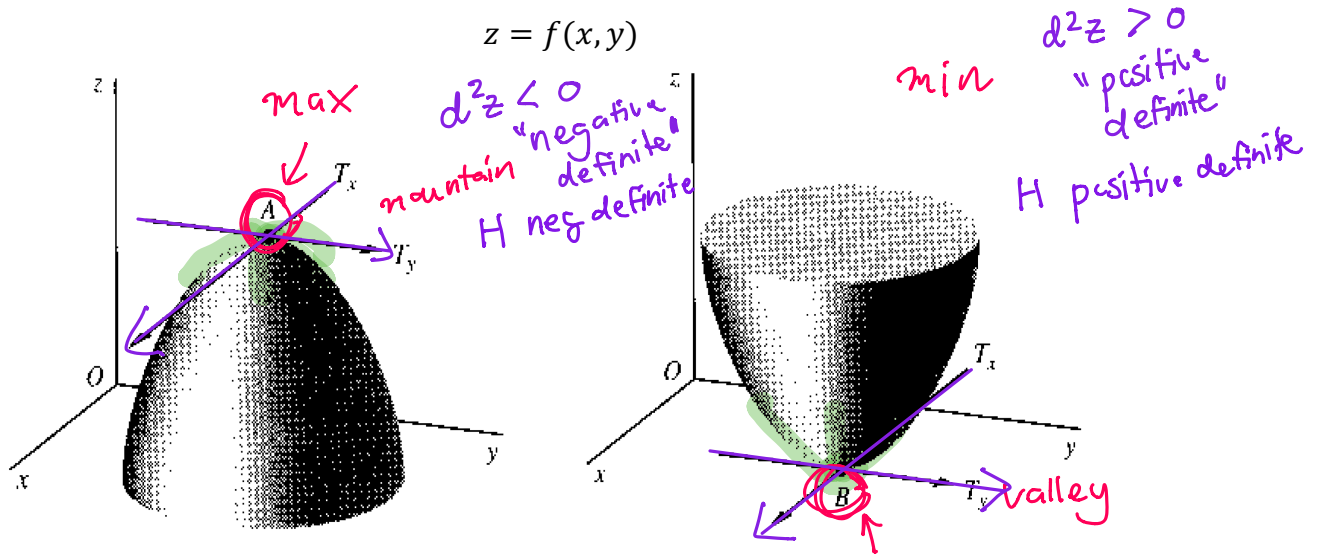
2<sup>nd</sup> order total differential of  $z$

$$d^2z = f''(x) \frac{dx^2}{> 0}$$



**Extreme Value of a Function of Two Variables**

$z = f(x, y)$



**First-Order Necessary Condition  $z = f(x, y)$**

$dz = 0$  for every  $dx$  and  $dy$

$dz = 0$

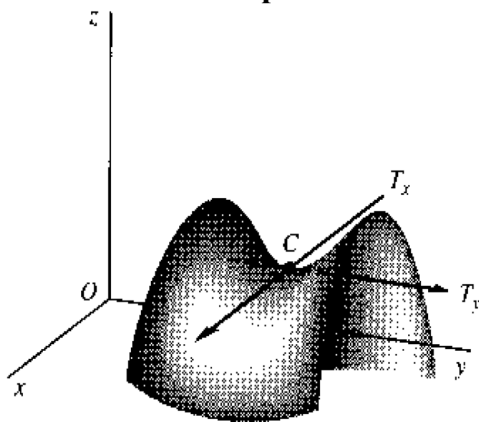
$\star dz = f_x dx + f_y dy = 0; dx, dy > 0$

$f_x = 0, f_y = 0$

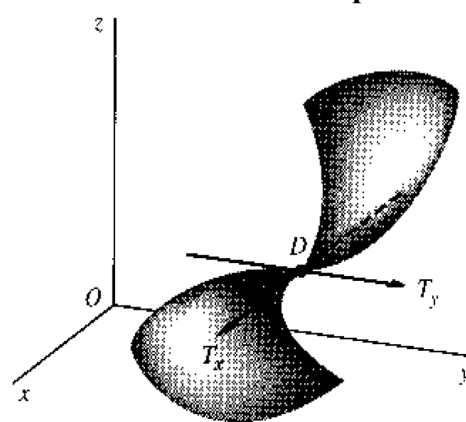
the slope of the tangent line  $T_x, T_y$  are zero

The first order necessary condition is necessary, but not sufficient to establish a maximum or a minimum.

**Saddle point**



**Inflection point**



**Second order sufficient condition**

The satisfaction of the first-order condition earmarks certain values of  $z$  as the stationary values of the objective function. If at a stationary value of  $z$  we find that  $d^2z$  is positive definite, i.e.,  $d^2z > 0$ , this will suffice to establish that value of  $z$  as a minimum. Analogously, the negative definiteness of  $d^2z$ ,  $d^2z < 0$ , is a sufficient condition for the stationary value to be a maximum.

$y = f(x_1, x_2, \dots, x_n)$   
 $H_{n \times n}$   $[f_{ij}]_{n \times n} \Rightarrow H_{n \times n}$

**Second-order total differential, Hessian Matrix and Definiteness of Hessian Matrix**

📖 About " $H_k$ "

$A_{n \times n}$

**Definition** : Let  $A$  be an  $n \times n$  matrix. A  $k \times k$  submatrix, deriving from deleting the last  $n - k$  columns and the last  $n - k$  rows of matrix  $A$ , can be called the  $k^{th}$  order leading principal submatrix of matrix  $A$ . The corresponding determinant of this  $k \times k$  submatrix is called the  $k^{th}$  order leading principal minor of matrix  $A$ .

The  $k^{th}$  order leading principal submatrix of matrix  $A$  is denoted by  $A_k$ .

The  $k^{th}$  order leading principal minor of matrix  $A$  is denoted by  $|A_k|$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{matrix} A_1 & A_2 & A_3 \\ & & 3 \times 3 \end{matrix}$$

Leading principal submatrices are:

The 1<sup>st</sup> order leading principal submatrix  $\Rightarrow$  delete the last  $3 - 1$  rows & columns

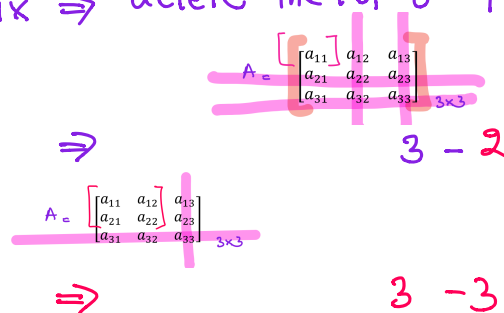
$A_1 = [a_{11}]$

2<sup>nd</sup>

$A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

3<sup>rd</sup>

$A_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = A$





### 📖 About "Definiteness of $H$ "

Let  $H$  be a symmetric matrix with dimension  $n \times n$

$d^2z > 0$  (a)  $H$  is **positive definite** if all  $n$  leading principal minors are positive

$$|H_1| > 0, |H_2| > 0, |H_3| > 0, \dots \quad \text{all positive}$$

$d^2z < 0$  (b)  $H$  is **negative definite** if  $n$  leading principal minor duly alternate in sign with the first one being negative.

$$|H_1| < 0, |H_2| > 0, |H_3| < 0, \dots \quad \text{starting with negative, alternate in sign}$$

If (a) or (b) is not met,  $H$  is indefinite.

Example: Find definiteness of matrix  $B = \begin{bmatrix} -3 & 4 \\ 4 & -6 \end{bmatrix}_{2 \times 2}$

$$B_1 = [-3], \quad |B_1| = -3 < 0$$

$$B_2 = B = \begin{bmatrix} -3 & 4 \\ 4 & -6 \end{bmatrix}, \quad |B_2| = 18 - 16 = 2 > 0$$

$B$  is neg. definite

Homework: Find definiteness of the following symmetric matrices ต่อไปนี้

(a)  $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

(e)  $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{bmatrix}$

(b)  $\begin{bmatrix} -3 & 4 \\ 4 & -5 \end{bmatrix}$

(f)  $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

(c)  $\begin{bmatrix} -3 & 4 \\ 4 & -6 \end{bmatrix}$

(g)  $\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 5 \\ 3 & 0 & 4 & 0 \\ 0 & 5 & 0 & 6 \end{bmatrix}$

**Second-Order Sufficient Condition (SOSC, SOC)**For the maximum of objective function  $z = f(x, y)$ the slope is decreasing ...  $d^2z < 0$ ,  $d^2z$  neg def. $\Rightarrow H$  is neg def.

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

$$|H_1| = f_{xx} < 0$$

$$|H_2| = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2 > 0$$

For the minimum of objective function  $z = f(x, y)$ the slope is increasing ...  $d^2z > 0$ ,  $d^2z$  positive def. $\Rightarrow H$  is positive def.

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

$$|H_1| = f_{xx} > 0$$

$$|H_2| = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2 > 0$$

Example: Find the maximum/minimum of function  $z = x + 2ey - e^x - e^{2y}$ 

$$\text{FOC: } \frac{\partial z}{\partial x} = 1 - e^x = 0 \Rightarrow x^* = 0$$

$$\frac{\partial z}{\partial y} = 2e - 2e^{2y} = 0 \Rightarrow e = e^{2y}$$

$$\text{note: } \frac{d}{dx} e^{kx} = ke^{kx} \quad 2y = 1 \Rightarrow y^* = \frac{1}{2}$$

$$\text{SOC: } f_{xx} = \frac{\partial f_x}{\partial x} = \frac{\partial (1 - e^x)}{\partial x} = -e^x$$

$$f_{xy} = f_{yx} = \frac{\partial f_x}{\partial y} = \frac{\partial (1 - e^x)}{\partial y} = 0$$

$$f_{yy} = \frac{\partial f_y}{\partial y} = \frac{\partial (2e - 2e^{2y})}{\partial y} = -4e^{2y}$$

$$H = \begin{bmatrix} -e^x & 0 \\ 0 & -4e^{2y} \end{bmatrix}$$

$$|H_1| = |-e^x| = -e^x < 0$$

$$|H_2| = 4e^{2+2y} = 4e^{0+2 \cdot \frac{1}{2}} = 4e > 0$$

$H$  is neg def.  $\therefore (x^*, y^*)$  gives  $z_{\max}$

$$z_{\max} = 0 + e - 1 - e = -1 \quad \#$$



**Objective functions with more three choice variables**

$$z = f(x_1, x_2, x_3)$$

F.O.N.C :

$$dz = 0$$

$$dz = f_1 dx_1 + f_2 dx_2 + f_3 dx_3 = 0; dx_1, dx_2, dx_3 > 0$$

$$f_1 = f_2 = f_3 = 0$$

$$f_1 = f_2 = f_3 = \dots = f_n = 0$$

We can use  $f_1 = f_2 = f_3 = 0$  to find the stationary values/critical values

S.O.S.C

$$H = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}_{3 \times 3}$$

All leading principal minor

$$|H_1| = |f_{11}|$$

$$|H_2| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}$$

$$|H_3| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$$

z is at the  $\begin{bmatrix} \text{maximum} \\ \text{minimum} \end{bmatrix}$  if  $\begin{cases} d^2z < 0 \\ d^2z > 0 \end{cases}$  if  $\begin{cases} |H_1| < 0, |H_2| > 0, |H_3| < 0 \\ |H_1| > 0, |H_2| > 0, |H_3| > 0 \end{cases}$  if  $\begin{cases} H \text{ is negative definite} \\ H \text{ is positive definite} \end{cases}$



### Objective functions with more $n$ –choice variables

$$z = f(x_1, x_2, x_3, \dots, x_n)$$

F.O.N.C :

$$dz = f_1 dx_1 + f_2 dx_2 + \dots + f_n dx_n = 0$$

$$\frac{\partial^2 f_i}{\partial x_i^2} = 0, \text{ for all } i, i = 1, 2, \dots, n$$

S.O.S.C:

$$\therefore |H| = \begin{vmatrix} f_{11} & f_{12} & \dots & f_{1n} \\ f_{21} & f_{22} & \dots & f_{2n} \\ \dots & \dots & \dots & \dots \\ f_{n1} & f_{n2} & \dots & f_{nn} \end{vmatrix}$$

All principal minors  $|H_1|, |H_2|, \dots, |H_n|$

If  $z$  is at is the maximum when  $d^2z < 0$ ,  $H$  is negative definite:

$$|H_1| < 0, |H_2| > 0, |H_3| < 0, |H_4| > 0, |H_5| < 0, \dots$$

If  $z$  is at is the minimum when  $d^2z > 0$ ,  $H$  is positive definite:

$$|H_1| > 0, |H_2| > 0, |H_3| > 0, |H_4| > 0, |H_5| > 0, \dots$$

**Homework:** Find extreme values of the following functions and check for the sufficient condition

$$z = 8x^3 + 2xy - 3x^2 + y^2 + 1$$

$$z = 2x_1^2 + x_1x_2 + 4x_2^2 + x_1x_3 + x_3^2 + 2$$

$$z = -3x_1^3 + 3x_1x_3 + 2x_2 - x_2^2 - 3x_3^2$$

$$z = x_1^2 + 3x_2^2 - 3x_1x_2 + 4x_2x_3 + 6x_3^2$$

$$z = x_1x_3 + x_1^2 - x_2 + x_2x_3 + x_2^2 + 3x_3^2$$

$$z = e^x + e^y + e^{w^2} - 2e^w - (x + y)$$

$$z = x^4 + x^2 - 6xy + 3y^2$$

$$w = x^2 + 6xy + y^2 - 3yz + 4z^2 - 10x - 5y - 21z$$

$$w = (x^2 + 2y^2 + 3z^2)e^{-(x^2 + y^2 + z^2)}$$

 **Summary of optimal conditions** 

The objective function:  $z = f(x, y)$

Condition	Maximum	Minimum
First-order necessary (FONC, FOC)	$f_x = f_y = 0$	$f_x = f_y = 0$
Second-order sufficient (SOSC, SOC)	$ H_1  = f_{xx} < 0$ $ H_2  = f_{xx}f_{yy} - f_{xy}^2 > 0$	$ H_1  = f_{xx} > 0$ $ H_2  = f_{xx}f_{yy} - f_{xy}^2 > 0$

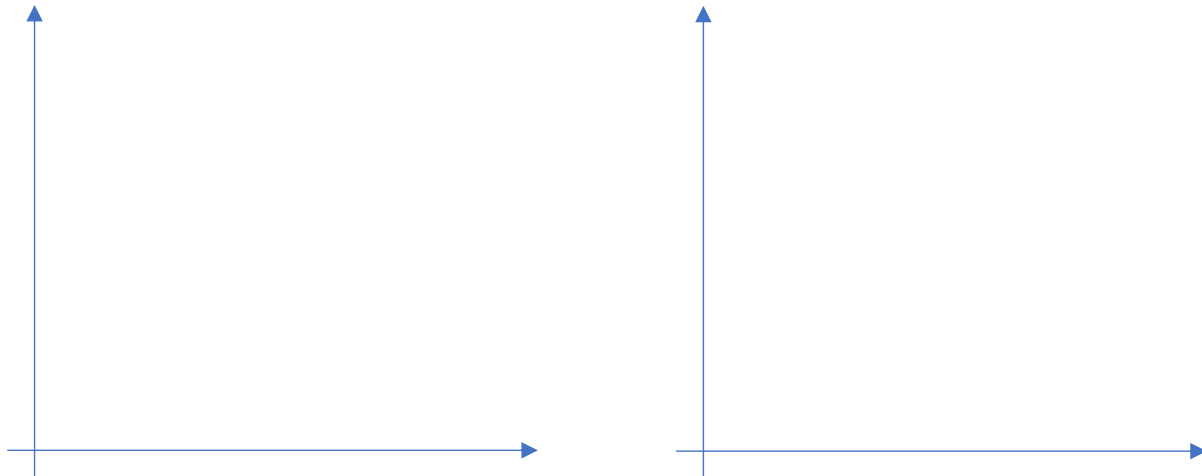
The objective function:  $z = f(x_1, x_2, \dots, x_n)$

Condition	Maximum	Minimum
First-order necessary (FONC, FOC)	$f_1 = f_2 = \dots = f_n = 0$	$f_1 = f_2 = \dots = f_n = 0$
Second-order sufficient (SOSC, SOC)	$H$ is negative definite. $(-1)^k  H_k  > 0$ $k = 1, 2, \dots, n$	$H$ is positive definite. $ H_k  > 0$ $k = 1, 2, \dots, n$



### Third- Degree Price Discrimination

Third-Degree Price Discrimination involves charging different prices to different segments of the market for the same good. This is to maximize profit within each segment of the market. For example, a theater may divide cinema goers into seniors, adults, and children, each paying a different price when seeing the same movie.



Total Revenue of a company is:

$$TR = R_1(Q_1) + R_2(Q_2) + R_3(Q_3),$$

$R_i$  is a total revenue function of market  $i$ .

Total cost of the company is:

$$C = C(Q)$$

$$Q = Q_1 + Q_2 + Q_3$$

$Q_i$  is quantity of product sold in market  $i$ .

Find the quantity and price in each market  $i$  that maximizes firm's profit.

**Step 1** State the objective Function: Profit function

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**Step 2** Find FONC and use FONC to find critical values

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$$MC = MR_1 = MR_2 = MR_3$$

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Conclusion: In the market with lower price elasticity of demand ( lower  $|\varepsilon_{di}|$ ), the maximizing-profit price in that market will be.....

**Step 3** Check SOSC whether  $Q_1^*, P_1^*, Q_2^*, P_2^*, Q_3^*, P_3^*$  indeed give the maximum profit

Hessian Matrix:

$$H = \begin{bmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{21} & \pi_{22} & \pi_{23} \\ \pi_{31} & \pi_{32} & \pi_{33} \end{bmatrix}$$





Check all leading principal minors

$$|H_1| =$$

$$|H_2| =$$

$$|H_3| =$$



### Competitive Firm Input Choices: Cobb-Douglas Technology

Let a firm in competitive market have Cobb-Douglas Production Function:

$$Q = L^\alpha K^\beta$$

Let wage be  $w$ , rent be  $r$  baht per unit.

Characteristics of the production function:

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**Step 1** State the objective Function: Profit function

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**Step 2** Find FONC and use FONC to find critical values

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### Multiproduct Firm

#### Multiproduct Firm in competitive market



Let a firm produce and sell two products. Each product is sold in competitive market, hence the firm is price taker. Total Revenue is:

$$TR =$$

$P_1$  is market price of good 1.

$P_2$  is market price of good 2.

$Q_1$  is quantity of good 1 sold in the market.

$Q_2$  is quantity of good 2 sold in the market.

Total cost of this firm is:

$$C = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$$

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Firm's Profit Function

$$\pi = TR - TC$$

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Choice Variables: .....





$TR =$ .....  
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Let the cost function of the monopolist be

$$C = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$$

Profit Function:

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Choice Variables: .....

Find FONC and use FONC to find critical values

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Check Second Order Sufficient Condition (SOSC)

$$H = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}$$

$$H = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$|H_1| =$$

$$|H_2| =$$

The second order differential of  $\pi$ ,  $d^2\pi$ , is .....definite. Hence, the profit is.....

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**Multiplant Firm Problem**

Consider a firm with many plants. Each plant  $i$  has total cost  $TC_i$ .

$$TC_i = C_i(q_i), i = 1, 2, \dots, n$$

$q_i$  is the level of output produced by factory  $i$ .

Assume that this firm sells its product in one market. Firm's total revenue function is:

$$TR = \dots\dots\dots$$

$$Q = \dots\dots\dots$$

If output market is competitive market,

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If output market is monopoly market,

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Firm's Profit Function

$$\pi = TR - TC$$

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Choice Variables: .....

*NOTE: Third-degree price discrimination vs. Multiplant firm*

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 .....  
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First Order Necessary Condition:

$$\pi_i = \dots \text{for all } i = 1, 2, \dots, n$$

SOSC:

$$H = \left[ \begin{array}{c} \\ \\ \\ \end{array} \right]$$

$$|H_1| = R'' - C_1'' < 0$$

$$|H_2| =$$

**Economic interpretation:**

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Example: Consider a monopolist firm with two plants:

Factory 1:  $C_1(Q_1) = 10Q_1^2$

Factory 2:  $C_2(Q_2) = 20Q_2^2$

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Firm's market demand is  $P = 700 - 5Q$ ,  $Q = Q_1 + Q_2$

Draw  $MC_1, MC_2, MC_T, AR, MR$  and indicate the maximizing level of output from each plant and the profit-maximizing price.





Check Second Order Sufficient Condition (SOSC)

$$H = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}$$

$$H = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$|H_1| =$$

$$|H_2| =$$

The second order differential of  $\pi$ ,  $d^2\pi$ , is .....definite. Hence, the profit is.....

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*Summary:*

*3<sup>rd</sup> price discrimination:*

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*Choosing levels of factor of production:*

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*Multiproduct:*

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*Multiplant:*

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