Quiz 2: Date: May 5, 2022 from 11.00-12.30

Question 1 (40 marks) Score.....................

Consider the Muliperiod model of consumption and portfolio choice. Let an individual in this economy has the utility function as follow:

$$\max_{C_s,\omega_s,\forall t} E_t\left[\sum_{s=t}^{T-1}\delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)+\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

Assume that there is no wage income $(y_t = 0\,\forall t)$ and a constant risk-free rate return asset , $R_{ft} = R_f$. Also asssume that n=1 and the return of a single risky asset, R_{rt},is independently and identically distributed over time. Denote the proportion of wealth invested in the risky asset at date t as ω_t.

Please read and answer the following questions carefully and completely.

Score.....................

Question 1.1 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-1, C^*_{T-1} and ω^*_{T-1}., and give an explicit expression for C^*_{T-1}

$$J(W_{T-1}, T-1) = \max_{C_{T-1}, \{w_{i,T-1}\}} U[C_{T-1}, T-1] + E_{T-1}[B(S_{T-1}R_{T-1}, T)]$$

Diff. WRT C_{T-1} : $U_C(C_{T-1}, T-1) - E_{T-1}[B_W[W_T, T)R_{T-1}] = 0$

" $\{w_{i,T-1}\}$: $E_{T-1}[B_W(W_T, T)(R_{i,T-1} - R_{f,T-1})] = 0$

$$U_C(C_{T-1}, T-1) = E_{T-1}[R_{f,T-1} B_W(W_T, T)$$

$$\frac{\delta^{T-1}(\cancel{1}-\gamma) C_{T-1}^{\cancel{1}-\gamma\cancel{-1}}}{1-\cancel{\gamma}} = \delta^T \frac{(\cancel{1}-\gamma) W_T^{(\cancel{1}-\gamma\cancel{-1})}}{(\cancel{1}-\gamma)}$$

$$\delta^{T-1} C_{T-1}^{-\gamma} = \delta^T W_T^{-\gamma}$$

$$C_{T-1}^{-\gamma} = \frac{\delta^{T-T+1}}{W_T^{\gamma}}$$

$$C^*_{T-1} = \sqrt[\gamma]{\frac{W_T^{\gamma}}{\delta}}$$

Score....................

Question 1.2 (10 marks) Solve for the form of J(W_{T-1},T-1).

$W = S + C$

util fn : $\dfrac{\delta^{T-1} \gamma \sqrt{\frac{W_T^{\gamma}}{\delta}}^{(1-\delta)}}{1-\gamma} + \delta^{T} \dfrac{W_T^{(1-\gamma)}}{1-\gamma}$

$J_W(W_{T-1}, T-1) = U_C(C^{*}_{T-1}, T-1)$

$= \delta^{T-1} \gamma \sqrt{\frac{W\gamma}{\delta}} + \dfrac{\delta^{T} \cancel{(1-\gamma)} (S_T + C_T)^{(1-\delta-1)} (2)}{\cancel{1-\gamma}}$

$= \delta^{T-1} \gamma \sqrt{\frac{W\delta}{\delta}} + \delta^{T} (S_T + C_T)^{-\gamma}$

Score....................

Question 1.3 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-2, C^*_{T-2} and ω^*_{T-2}., and give an explicit expression for C^*_{T-2}

$$J(W_{T-2}, T-2) = \max_{C_{T-2}, \{w_{i,T-2}\}} U(C_{T-2}, T-2) + E_{T-2}[U(C_{T-1}, T-1) + B(W_T, T)]$$

Diff. WRT C_{T-2} : $U_C(C_{T-2}, T-2) - E_{T-2}[B_W[W_{T-1}, T-1)R_{T-2}] = 0$

" $\{w_{i,T-2}\}$: $E_{T-2}[B_W(W_{T-1}, T-1)(R_{i,T-2}, R_{f,T-2})] = 0$

$$U_C(C_{T-2}, T-2) = E_{T-2}[R_{f,T-2} B_W(W_{T-1}, T-1)]$$

$$\frac{\delta^{T-2}\cancel{(1-\gamma)}\, C_{T-2}^{\cancel{1}-\gamma\cancel{-1}}}{\cancel{1-\gamma}} = \delta^{T-1}\frac{\cancel{(1-\gamma)}\, W_{T-1}^{(\cancel{1}-\gamma\cancel{-1})}}{\cancel{(1-\gamma)}}$$

$$\delta^{T-1} C_{T-2}^{-\gamma} = \delta^{T-2} W_{T-1}^{-\gamma}$$

$$C_{T-2}^{-\gamma} = \delta^{T-2-T+1} W_{T-1}^{-\gamma}$$

$$C_{T-2} = \sqrt[\gamma]{\frac{1}{\delta W_{T-1}^{\gamma}}}$$

Score.....................

Question 1.4 (10 marks) Solve for the form of J(W_{T-2},T-2).Based on the pattern for T-1 and T-2, provide expressions for the optimal consumption and portfolio weight at any date T-t, t=1,2,3,....

$W = S + C$

util fn :
$$\frac{\delta^{T-2} \sqrt[\gamma]{\frac{1}{\delta W_{T-1}^{\gamma}}}}{1-\gamma} + \delta^{T-1} \frac{W_{T-1}^{(1-\gamma)}}{1-\gamma}$$

$$J_W(W_{T-2}, T-2) = U_C(C^*_{T-2}, T-2)$$
$$= \delta^{T-2} \sqrt[\gamma]{\frac{1}{\delta W_{T-1}^{\gamma}}} + \delta^{T-1} \frac{\cancel{(1-\gamma)}(S_{T-1}+C_{T-1})^{(1-\gamma-1)}\ (2)}{\cancel{1-\gamma}}$$
$$= \delta^{T-2} \sqrt[\gamma]{\frac{1}{\delta W_{T-1}^{\gamma}}} + \delta^{T-1}(S_{T-1}+C_{T-1})^{-\gamma}$$

Cont.

For any t

$$U_C(C^*_t, t) = J_W(W_t, t)$$