

Chapter 1 Limits and Continuity

Section 1.1 Limits

Consider the rational function $f(x) = \frac{x^2 - 1}{x - 1}$. Clearly we cannot use $x = 1$.

We want to know what happens if our x values approach $x = 1$. In this case

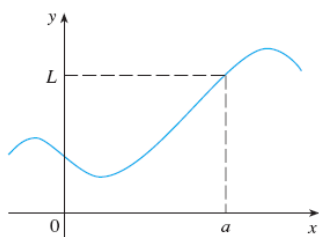
$x < 1$	$f(x)$	$x > 1$	$f(x)$
0.9	1.9	1.01	2.01
0.99	1.99	1.001	2.001
0.999	1.999	1.0001	2.0001

From the tables, we can see that if our x values approach $x = 1$, then our $f(x)$ (or y) values approach 2. This idea is expressed using the limit. We write $\lim_{x \rightarrow 1} f(x) = 2$ and we read “The limit of $f(x)$ as x approaches one is two”.

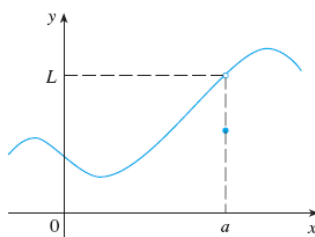
In general,

$$\lim_{x \rightarrow a} f(x) = L$$

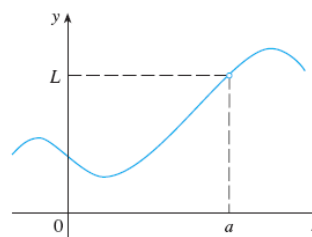
In other words, we can make our y values as close to L as we wish, simply by choosing x values close enough to a .



(a)



(b)



(c)

In all of these cases, $\lim_{x \rightarrow a} f(x) = L$.

▪ One sided Limits

Consider the split definition function

$$f(x) = \begin{cases} x^2 - 1 & \text{if } x \geq 0 \\ \frac{1}{2}x + 2 & \text{if } x < 0 \end{cases}$$

Find $\lim_{x \rightarrow 0} f(x)$.

We have $\lim_{x \rightarrow 0} f(x)$ does not exist.

However,

$$\lim_{x \rightarrow 0^+} f(x) = -1 \quad \text{and} \quad \lim_{x \rightarrow 0^-} f(x) = 2.$$

We write

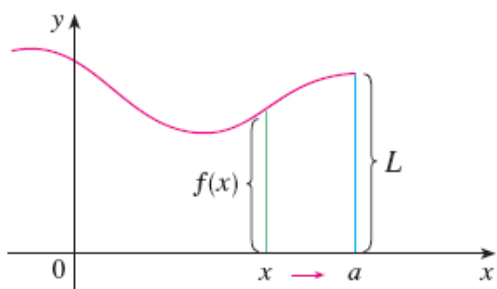
$$\lim_{x \rightarrow a^-} f(x) = L$$

“The left – hand limit of $f(x)$ as x approaches a (OR the limit of $f(x)$ as x approaches a from the left) is equal to L .” If we can make the value of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x less than a .

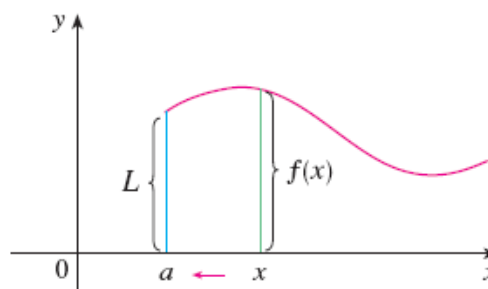
Similarly, we write

$$\lim_{x \rightarrow a^+} f(x) = L$$

“The right – hand limit of $f(x)$ as x approaches a (OR the limit of $f(x)$ as x approaches a from the right) is equal to L .” If we can make the value of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x greater than a .

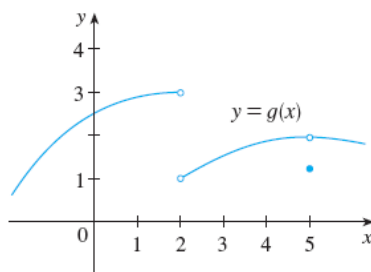


(a) $\lim_{x \rightarrow a^-} f(x) = L$



(b) $\lim_{x \rightarrow a^+} f(x) = L$

Example 1: Given $g(x)$



Find

(a) $\lim_{x \rightarrow 2^-} g(x)$

(b) $\lim_{x \rightarrow 2^+} g(x)$

(c) $\lim_{x \rightarrow 2} g(x)$

(d) $\lim_{x \rightarrow 5^-} g(x)$

(e) $\lim_{x \rightarrow 5^+} g(x)$

(f) $\lim_{x \rightarrow 5} g(x)$

(g) $f(5)$

Example 2: Graph of $g(x)$ is given as in Example 1. Determine the values of a for which $\lim_{x \rightarrow a} g(x)$ exists.

Example 3: Sketch the graph of an example of a function f that satisfies all of the given conditions.

$\lim_{x \rightarrow 3^+} f(x) = 4$

$\lim_{x \rightarrow 3^-} f(x) = 2$

$\lim_{x \rightarrow -2} f(x) = 2$

$f(3) = 3$

$f(-2) = 1$

Section 1.2 Computing Limits

Limit Laws: Suppose that c is a constant and the limits

$$\lim_{x \rightarrow a} f(x) \quad \text{and} \quad \lim_{x \rightarrow a} g(x)$$

exist. Then

$$1. \lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

$$2. \lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$$

$$3. \lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x)$$

$$4. \lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$$

$$5. \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \quad \text{where} \quad \lim_{x \rightarrow a} g(x) \neq 0$$

$$6. \lim_{x \rightarrow a} [f(x)]^n = [\lim_{x \rightarrow a} f(x)]^n \quad \text{where } n \text{ is a positive integer}$$

$$7. \lim_{x \rightarrow a} c = c$$

$$8. \lim_{x \rightarrow a} x = a$$

$$9. \lim_{x \rightarrow a} x^n = a^n \quad \text{where } n \text{ is a positive integer}$$

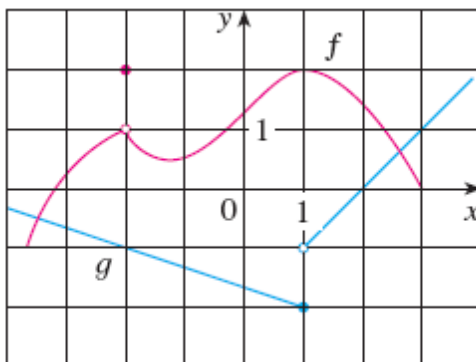
$$10. \lim_{x \rightarrow a} \sqrt[n]{x} = \sqrt[n]{a} \quad \text{If } n \text{ is even, we assume that } a > 0$$

$$11. \lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)} \quad \text{If } n \text{ is even, we assume that } \lim_{x \rightarrow a} f(x) > 0$$

Direct Substitution Property: If f is a polynomial or rational function and a is in the domain of f , then

$$\lim_{x \rightarrow a} f(x) = f(a)$$

Example 1:



Use the Limit Laws and the graphs of f and g to evaluate the following limits, if they exist.

(a) $\lim_{x \rightarrow -2} [f(x) + 5g(x)]$

(b) $\lim_{x \rightarrow 1} [f(x)g(x)]$

(c) $\lim_{x \rightarrow 2} \frac{f(x)}{g(x)}$

Example 2: Find the following limits.

$$(a) \lim_{x \rightarrow 1} \frac{3x + \sqrt{x+3}}{x-2}$$

$$(b) \lim_{x \rightarrow 2} \frac{2x - 8 + x^2}{x-2}$$

$$(c) \lim_{x \rightarrow -1} x + 5 - \frac{(x+1)^2}{x^2 - x - 2}$$

$$(d) \lim_{x \rightarrow 2} \frac{\sqrt{x-1} - 1}{x-2}$$

Note: Some limits are best calculated by first finding the left – and right – hand limits.

Theorem:

$$\lim_{x \rightarrow a} f(x) = L \quad \text{if and only if} \quad \lim_{x \rightarrow a^-} f(x) = L = \lim_{x \rightarrow a^+} f(x)$$

Example 3: Prove that $\lim_{x \rightarrow 0} \frac{|x|}{x}$ does not exist.

Example 4: Find the limit, if it exists.

$$(a) \lim_{x \rightarrow 2} \frac{|x - 2|}{x - 2}$$

$$(b) \lim_{x \rightarrow 1.5} \frac{2x^2 - 3x}{|2x - 3|}$$

Example 5: If

$$f(x) = \begin{cases} \sqrt{x+6} & \text{if } x > 3 \\ x^2 & \text{if } x < 3 \end{cases}$$

Determine whether $\lim_{x \rightarrow 3} f(x)$ exists.

Theorem: If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a , then

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x)$$

The Squeeze Theorem: If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

then

$$\lim_{x \rightarrow a} g(x) = L.$$

Example 6: If $1 \leq f(x) \leq x^2 + 2x + 2$ for all x , find $\lim_{x \rightarrow -1} f(x)$.

Example 7: Prove that $\lim_{x \rightarrow 0} x^2 \cos \frac{2}{x} = 0$

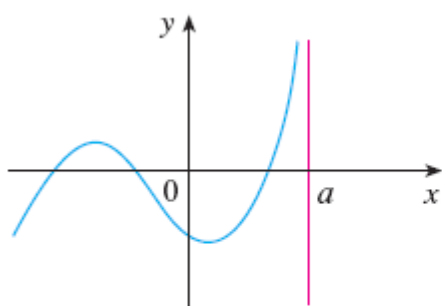
Section 1.3 Limits at Infinity

▪ Vertical Asymptote

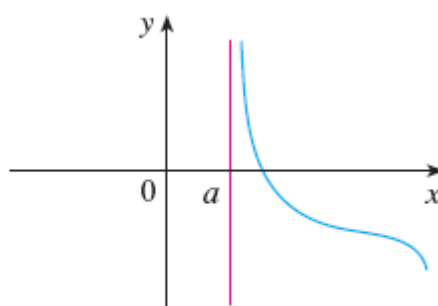
Definition: The line $x = a$ is called a *vertical asymptote* of the curve $y = f(x)$ if at least one of the following statements is true:

$$\begin{array}{lll} \lim_{x \rightarrow a^-} f(x) = \infty & \lim_{x \rightarrow a^-} f(x) = \infty & \lim_{x \rightarrow a^+} f(x) = \infty \\ \lim_{x \rightarrow a^-} f(x) = -\infty & \lim_{x \rightarrow a^-} f(x) = -\infty & \lim_{x \rightarrow a^+} f(x) = -\infty \end{array}$$

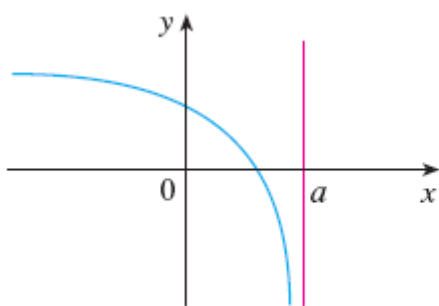
For example,



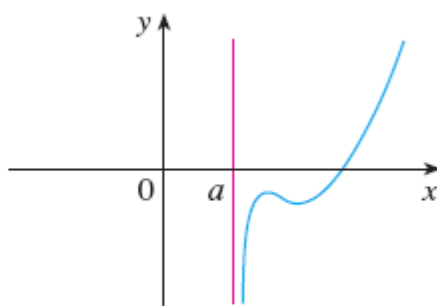
(a) $\lim_{x \rightarrow a^-} f(x) = \infty$



(b) $\lim_{x \rightarrow a^+} f(x) = \infty$



(c) $\lim_{x \rightarrow a^-} f(x) = -\infty$



(d) $\lim_{x \rightarrow a^+} f(x) = -\infty$

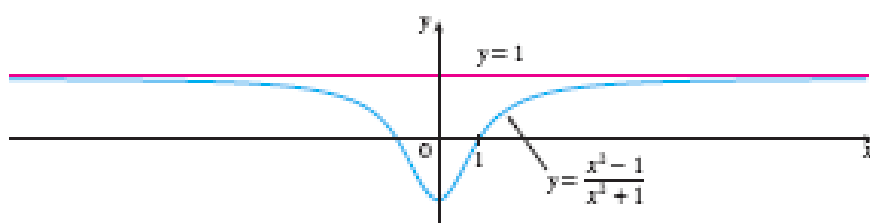
▪ Horizontal Asymptote

Definition: The line $y = L$ is called a *horizontal asymptote* of the curve $y = f(x)$ if either

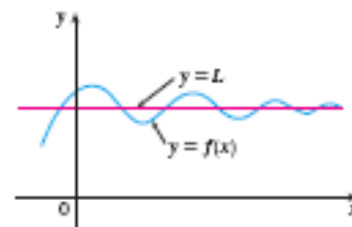
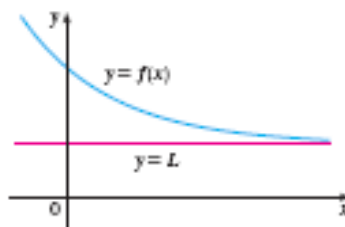
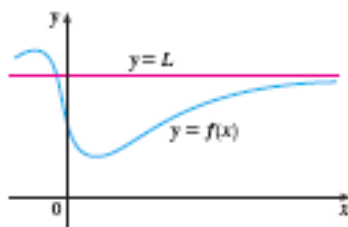
$$\lim_{x \rightarrow \infty} f(x) = L \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = L$$

Example: Consider $f(x) = \frac{x^2 - 1}{x^2 + 1}$

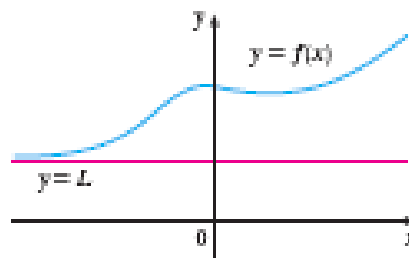
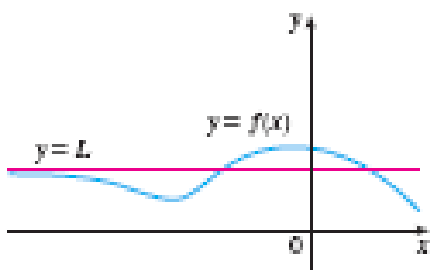
x	0	± 1	± 2	± 3	± 4	± 10	± 50	± 100	± 1000
$f(x)$	-1	0	0.6	0.8	0.882	0.98	0.9992	0.9998	0.999998



Examples illustrating $\lim_{x \rightarrow \infty} f(x) = L$:



Examples illustrating $\lim_{x \rightarrow -\infty} f(x) = L$:



Theorem: If $r > 0$ is a rational number, then

$$\lim_{x \rightarrow \infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x , then

$$\lim_{x \rightarrow -\infty} \frac{1}{x^r} = 0$$

Example 2: Find the horizontal asymptotes and vertical asymptotes of the graph of the function.

$$(1) f(x) = \frac{x^2 - x}{x^2 - 1}$$

$$(2) f(x) = \frac{x^2 - x}{x - 1}$$

▪ Infinite Limits at Infinity

The notation

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

is used to indicate that the values of $f(x)$ become large. Similar meaning are attached to

$$\lim_{x \rightarrow \infty} f(x) = -\infty \quad \lim_{x \rightarrow -\infty} f(x) = \infty \quad \lim_{x \rightarrow -\infty} f(x) = -\infty$$

Example: Find

$$(1) \lim_{x \rightarrow \infty} 2x^3$$

$$(2) \lim_{x \rightarrow -\infty} 2x^3$$

$$(3) \lim_{x \rightarrow -\infty} 3x^2$$

$$(4) \lim_{x \rightarrow \infty} \frac{x^2 - x}{x - 1}$$

$$(5) \lim_{x \rightarrow -\infty} \frac{x^2 + x}{x + 3}$$

$$(6) \lim_{x \rightarrow \infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1}$$

$$(7) \lim_{x \rightarrow -\infty} \left(x + \sqrt{x^2 + 2x} \right)$$

$$(8) \lim_{x \rightarrow -\infty} \frac{x}{\sqrt{x^2 + 4}}$$

$$(9) \lim_{x \rightarrow -\infty} (x^4 + x^5)$$

Section 1.4 Limits of Trigonometric Functions

Lemma1: $\lim_{x \rightarrow 0} \sin x = 0$

Lemma2: $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

Lemma3: For any real number $k \neq 0$, $\lim_{x \rightarrow 0} \frac{\sin kx}{x} = k$

Note : From Lemma 2 and Lemma 3, we have

$$\lim_{x \rightarrow 0} \frac{x}{\sin x} = 1 \quad \text{and} \quad \lim_{x \rightarrow 0} \frac{x}{\sin kx} = \frac{1}{k}$$

Example 1: Find the following limits.

(1) $\lim_{u \rightarrow 0} \frac{\sin(3u^2)}{u^2}$

(2) $\lim_{h \rightarrow 0} \frac{\sin(2h)}{\tan(h)}$

(3) $\lim_{x \rightarrow 0} \frac{2x^2 + \sin^2(3x)}{x^2}$

Section 1.5 Continuity

Notice that the limit of a function as x approaches a can often be found simply by calculating the value of the function at a . Functions with this property are called continuous at a .

Definition: A function is said to be continuous at a if

- (1) $f(a)$ is defined (that is, a is in the domain of f)
- (2) $\lim_{x \rightarrow a} f(x)$ exists.
- (3) $\lim_{x \rightarrow a} f(x) = f(a)$.

Example:

Example 1: At which number is f discontinuous? Why?

Example 2: Given

$$f(x) = \begin{cases} \frac{x^2 - 1}{x - 1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$$

Is f continuous at 1?

Example 3: Use the definition of continuity and the properties of limits to show that the function is continuous at the given number.

$$(a) f(x) = (x^3 + x)^2, \quad a = 1$$

$$(b) f(x) = \frac{x-1}{x^2+3}, \quad a = -1$$

Example 4: Where are each of the following functions discontinuous?

$$(a) f(x) = \frac{x^2 + x - 2}{x - 1}$$

$$(b) f(x) = \begin{cases} \frac{x^2 + x - 2}{x - 1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$$

Example 5: Find the value of c that makes f continuous on $(-\infty, \infty)$.

$$f(x) = \begin{cases} cx + 1 & \text{if } x \leq 3 \\ cx^2 - 1 & \text{if } x > 3 \end{cases}$$

Theorem: If f and g are continuous at a and c is a constant, then the following functions are also continuous at a :

1. $f + g$
2. $f - g$
3. cf
4. fg
5. f/g if $g(a) \neq 0$

Theorem: The following types of functions are continuous at every number in their domains:

- Polynomials
- Root functions
- Rational functions
- Trigonometric functions

Example 7: On what intervals is each function continuous?

(1) $f(x) = x^2 - 2x + 1$

(2) $g(x) = \frac{5x}{x+1}$

(3) $f(x) = \sqrt{x} + \frac{x+1}{x-1} - \frac{x+1}{x^2+1}$

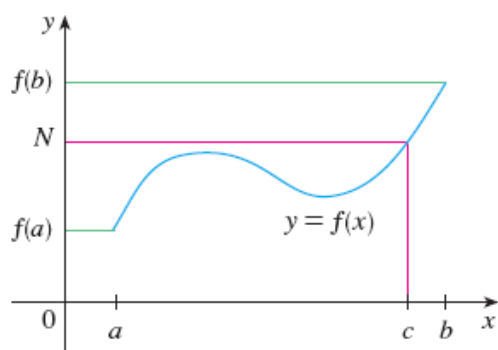
Example 8: Given

$$f(x) = \begin{cases} 1+x^2 & \text{if } x \leq 0 \\ 2-x & \text{if } 0 < x \leq 2 \\ (x-2)^2 & \text{if } x > 2 \end{cases}$$

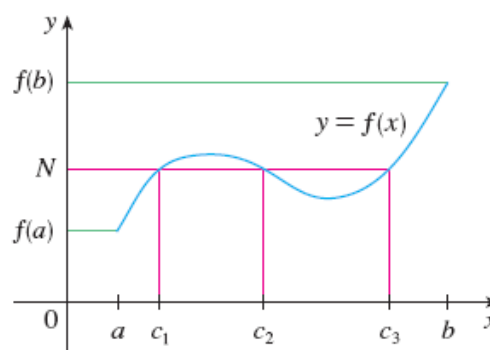
Find the numbers at which f is discontinuous.

The Intermediate Value Theorem

Suppose that f is continuous on the closed interval $[a,b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a,b) such that $f(c) = N$.



(a)



(b)

Example 10: Use the Intermediate Value Theorem to show that there is a root of the equation $x^4 + x - 3 = 0$ between 1 and 2.