Solution: Assignment 6

1. Find all *vertical*, *horizontal*, *slant* asymptotes (if any) of the following relations.

(a)
$$r = \left\{ (x,y) \in X \times Y \middle| \quad yx^3 + y = -2x^3 - 3x + 5 \right\}.$$

(b)
$$s = \left\{ (x,y) \in X \times Y \middle| \quad 2x^2 + x - xy - y = 5 \right\}.$$

(c)
$$f = \left\{ (x,y) \in X \times Y \middle| \quad y = \frac{6x^3 - 1}{18 - 2x^2} \right\}.$$

Solution:

(a)
$$r = \left\{ (x,y) \in X \times Y \middle| \quad yx^3 + y = -2x^3 - 3x + 5 \right\}.$$

For $(x, y) \in r$,

$$yx^3 + y = -2x^3 - 3x + 5 \quad \Rightarrow \quad y = \frac{-2x^3 - 3x + 5}{x^3 + 1} \ \Rightarrow r(x) = \frac{-2x^3 - 3x + 5}{x^3 + 1}.$$

Let $p(x) = -2x^3 - 3x + 5$ and $q(x) = x^3 + 1$ so that $r(x) = p(x)/q(x)$. Let m be the degree of the numerator $p(x) - 2x^3 - 3x + 5$ and n be the degree of the numerator $q(x) = x^3 + 1$. Then $m = 3$ and $n = 3$.

(i) Since $m = n$, we can have horizontal asymptote(s). In particular, consider $\lim_{x\to\infty} r(x)$ and $\lim_{x\to-\infty} r(x)$:

$$\lim_{x\to\infty} r(x) = \lim_{x\to\infty} \frac{-2x^3 - 3x + 5}{x^3 + 1} = \lim_{x\to\infty} \frac{-2x^3}{x^3} = -2$$

and

$$\lim_{x\to-\infty} r(x) = \lim_{x\to-\infty} \frac{-2x^3 - 3x + 5}{x^3 + 1} = \lim_{x\to-\infty} \frac{-2x^3}{x^3} = -2.$$

That is, $\boxed{y = -2}$ is the horizontal asymptote.

(ii) For the vertical asymptote, we need to first look for the zeros of the denominator $q(x) = x^3+1$. Since, $q(x) = x^3 + 1 > 0$ for all $x \in \mathbb{R}$, then there is no vertical asymptote.

(iii) Slant asymptote occurs when $m = n + 1$ which is not true in this case. Hence, there is no slant asymptote.

From (i), (ii), and (iii), there is only one horizontal asymptote: $\boxed{y = -2}$ for the graph $y = r(x)$. ∎

(b)

$$s = \left\{ (x,y) \in X \times Y \,\middle|\, 2x^2 + x - xy - y = 5 \right\}.$$

For $(x, y) \in s$,

$$2x^2 + x - xy - y = 5 \quad \Rightarrow \quad xy + y = 2x^2 + x - 5 \quad \Rightarrow \quad y = \frac{2x^2 + x - 5}{x+1}.$$

Let $p(x) = 2x^2 + x - 5$ and $q(x) = x + 1$ so that $s(x) = p(x)/q(x)$. Let m be the degree of the numerator $p(x) = 2x^2 + x - 5$ and n be the degree of the numerator $q(x) = x + 1$. Then $m = 2$ and $n = 1$.

(i) Since $m > n$, we cannot have horizontal asymptote.

In particular, $\lim\limits_{x\to\infty} s(x)$ and $\lim\limits_{x\to-\infty} s(x)$ are not going to be equal to any fixed constant:

$$\lim_{x\to\infty} s(x) = \lim_{x\to\infty} \frac{2x^2 + x - 5}{x+1} = \lim_{x\to\infty} \frac{2x^2}{x} = \lim_{x\to\infty} 2x = \infty$$

and

$$\lim_{x\to-\infty} s(x) = \lim_{x\to-\infty} \frac{2x^2 + x - 5}{x+1} = \lim_{x\to-\infty} \frac{2x^2}{x} = \lim_{x\to-\infty} 2x = -\infty.$$

(ii) For the vertical asymptote, we need to look for the zeros of the denominator $q(x) = x + 1$. Since, $q(x) = x + 1 > 0$ for $x = -1$ and $p(-1) = 2 - 1 - 5 \neq 0$ (or $x + 1$ is not the factor of $p(x)$), then a vertical asymptote is $x = -1$.

(iii) Slant asymptote occurs when $m = n + 1$ which is true in this case. Hence, there is a slant asymptote.

$$\begin{array}{rl}
 & 2x-1 \\ \cline{2-2}
x+1 & \big)\, 2x^2+x-5 \\
 & 2x^2+2x \\ \cline{2-2}
 & -x-5 \\
 & -x-1 \\ \cline{2-2}
 & -4
\end{array}$$

That is,

$$s(x) = \frac{2x^2 + x - 5}{x+1} = (2x - 1) + \frac{-4}{x+1}$$

and the slant asymptote for $y = s(x)$ is $\boxed{y = 2x - 1}$. We can check this by using the definition as follows:

$$\lim_{x\to\infty} [s(x) - (2x-1)] = \lim_{x\to\infty} \left[(2x-1) + \frac{-4}{x+1} - (2x-1) \right] = \lim_{x\to\infty} \frac{-4}{x+1} = 0$$

and similarly we can show that $\lim\limits_{x\to-\infty} [s(x) - (2x-1)] = 0$.

From (i), (ii), and (iii), there are a **vertical** asymptote $\boxed{x = -1}$ and a **slant asymptote** $\boxed{x = 2x - 1}$ for the graph $y = s(x)$. ■

(c)

$$f = \left\{ (x, y) \in X \times Y \middle| \quad y = \frac{6x^3 - 1}{18 - 2x^2} \right\}.$$

For $(x, y) \in f$,

$$y = \frac{6x^3 - 1}{18 - 2x^2} \quad \Rightarrow \quad f(x) = \frac{6x^3 - 1}{18 - 2x^2}.$$

Let $p(x) = 6x^3 - 1$ and $q(x) = 18 - 2x^2$ so that $f(x) = p(x)/q(x)$. Let m be the degree of the numerator $p(x) = 6x^3 - 1$ and n be the degree of the numerator $q(x) = 18 - 2x^2$. Then $m = 3$ and $n = 2$.

(i) Since $m > n$, we cannot have horizontal asymptote.

In particular, $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ are not going to be equal to any fixed constant:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{6x^3 - 1}{18 - 2x^2} = \lim_{x\to\infty} \frac{6x^3}{-2x^2} = \lim_{x\to\infty} -3x = -\infty$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{6x^3 - 1}{18 - 2x^2} = \lim_{x\to-\infty} \frac{6x^3}{-2x^2} = \lim_{x\to-\infty} -3x = \infty.$$

Hence, **there is no** a horizontal asymptote.

(ii) For the vertical asymptote, we need to look for the zeros of the denominator $q(x) = 18 - 2x^2$. Notice that

$$q(x) = 2(9 - x^2) = 2(3 - x)(3 + x) = 0 \text{ for } x = -3, 3.$$

Also, $p(-3) = 6(-3)^3 - 1 \neq 0$ and $p(3) = 6(3)^3 - 1 \neq 0$ (i.e. neither $x - 3$ nor $x + 3$ is a factor of $p(x)$). Therefore there are two vertical asymptotes: $\boxed{x = -3 \text{ and } x = 3}$.

(iii) Slant asymptote occurs when $m = n + 1$ which is true in this case. Hence, there is a slant asymptote.

$$\begin{array}{r|l} & -3x \\ \hline -2x^2 + 18 & 6x^3 - 1 \\ & 6x^3 - 54x \\ \hline & \\ & 54x - 1 \end{array}$$

That is,

$$f(x) = \frac{6x^3 - 1}{-2x^2 + 18} = (-3x) + \frac{54x - 1}{-2x^2 + 18}$$

and the slant asymptote for $y = s(x)$ is $\boxed{y = -3x}$. We can check this by using the definition as follows:

$$\lim_{x\to\infty} [f(x) - (-3x)] = \lim_{x\to\infty} \left[(-3x) + \frac{54x - 1}{-2x^2 + 18} - (-3x) \right] = \lim_{x\to\infty} \frac{54x - 1}{-2x^2 + 18} = 0$$

and similarly we can show that $\lim_{x\to-\infty} [f(x) - (-3x)] = 0$.

From (i), (ii), and (iii), there are two **vertical** asymptotes $\boxed{x = -3,\ x = 3}$ and a **slant asymptote** $\boxed{x = -3x}$ for the graph $y = f(x)$.

■

2. [**CORRECTED**] Consider the relation f defined from X to Y, $X, Y \subseteq \mathbb{R}$,

$$f = \left\{ (x,y) \in X \times Y \middle| \quad y = \frac{x^3}{x^2-4} \right\}.$$

Hint: For $f(x) = \frac{x^3}{x^2-4}$, the first and the second derivatives are given, respectively, by

$$f'(x) = \frac{x^4 - 12x^2}{(x^2-4)^2}, \qquad \text{and} \qquad f''(x) = \frac{8x^3 + 96x}{(x^2-4)^3}.$$

(a) Find the domain of f.

(b) Find x-intercepts and y-intercepts (if any).

(c) Determine the symmetry of f.

(d) Find the horizontal and vertical asymptotes for f (if any).

(e) Find the critical number of f. Determine the intervals on which f is increasing and decreasing. Determine the extrema (maximum and minimum) of f.

(f) Determine the intervals on which f is concave up and concave down. Find the points of inflection (if any).

(g) Sketch the curve of f.

Solution:

(a) Since $y = \frac{x^3-1}{x^2-4}$, we can also write $f(x) = \frac{x^3-1}{x^2-4}$. The domain of f , which consists of possible values of x that make $x^2 - 4 \neq 0$.

$$x^2 - 4 = 0 \quad \Leftrightarrow \quad (x-2)(x+2) = 0 \quad \Leftrightarrow \quad x = -2, 2$$

Hence, the domain is $\mathbb{R} - \{-2, 2\}$.

(b) x-intercepts: setting $y = 0$ gives $0 = \frac{x^3}{x^2-1}$, which occurs if and only if

$$x^3 = 0 \quad \Leftrightarrow \quad x = 0.$$

So x-intercepts is $(0,0)$.
To find y-intercepts, setting $x = 0$ gives $y = \frac{0}{0-4} = 0$. So y-intercepts is also $(0,0)$.

(c) Determine the symmetry of f.
Notice that f is a function and can be written as $f(x) = \frac{x^3}{x^2-4}$. Hence, it is not symmetric about the x-axis. We will look at the symmetry by using $f(-x)$.

$$f(-x) = \frac{(-x)^3}{(-x)^2 - 4} = -\frac{x^3}{x^2-4} = -f(x).$$

Since $f(-x) = -f(x)$, f is an odd function and it is symmetric about the origin.
Note that $f(-x) \neq f(x)$, it is not symmetric about the y-axis.

(d) Find the horizontal and vertical asymptotes for f (if any).

Horizontal asymptotes:

Let $p(x) = x^3$ and $q(x) = x^2 - 4$ so that $f(x) = p(x)/q(x)$. Let m be the degree of the numerator $p(x) = x^3$ and n be the degree of the numerator $q(x) = x^2 - 4$. Then $m = 3$ and $n = 2$.

(i) Since $m > n$, we cannot have horizontal asymptote.

In particular, $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ are not going to be equal to any fixed constant:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^3}{x^2-4} = -\infty \quad \text{and} \quad \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^3}{x^2-4} = \infty$$

So, there is **no** horizontal asymptote.

Vertical asymptotes: For the vertical asymptote, we need to look for the zeros of the denominator $q(x) = x^2 - 4$. Notice that

$$q(x) = x^2 - 4 = (x-2)(x+2) = 0 \text{ for } x = -2, 2.$$

Also, $p(-2) = (-2)^3 \neq 0$ and $p(2) = (2)^3 \neq 0$ (i.e. neither $x - 2$ nor $x + 2$ is a factor of $p(x)$). Therefore there are two vertical asymptotes: $\boxed{x = -2 \text{ and } x = 2}$. This can be shown by the definition as follows.

$$\lim_{x\to-2^-} f(x) = \lim_{x\to-2^-} \frac{x^3}{x^2-4} = -\infty \qquad \frac{(-)}{(+)}$$

$$\lim_{x\to-2^+} f(x) = \lim_{x\to-2^+} \frac{x^3}{x^2-4} = +\infty \qquad \frac{(-)}{(-)}$$

$$\lim_{x\to2^-} f(x) = \lim_{x\to2^-} \frac{x^3}{x^2-4} = -\infty \qquad \frac{(+)}{(-)}$$

$$\lim_{x\to2^+} f(x) = \lim_{x\to2^+} \frac{x^3}{x^2-4} = +\infty \qquad \frac{(+)}{(+)}$$

(e) Find the critical number of f.

$$f'(x) = \frac{x^4 - 12x^2}{(x^2-4)^2}$$

The critical number x occurs when $f'(x) = 0$ or $f'(x)$ does not exist.

$$f'(x) = 0 \quad \Leftrightarrow \quad x^4 - 12x^2 = x^2(x^2-12) = x^2(x-2\sqrt{3})(x+2\sqrt{3}) = 0 \quad \Leftrightarrow \quad x = 0, -2\sqrt{3}, 2\sqrt{3}.$$

(f) Determine the intervals on which f is increasing and decreasing. Determine the relative extrema (maximum and minimum) of f.

From

$$f'(x) = \frac{x^4 - 12x^2}{(x^2-4)^2} = \frac{x^2(x-2\sqrt{3})(x+2\sqrt{3})}{(x^2-4)^2}$$

Notice that the denominator is always positive.

- Decreasing: $f'(x) < 0$ when $x^2(x-2\sqrt{3})(x+2\sqrt{3}) < 0$.
- Increasing: $f'(x) > 0$ when $x^2(x-2\sqrt{3})(x+2\sqrt{3}) > 0$.

	$x < -2\sqrt{3}$	$-2\sqrt{3} < x < 0$	$0 < x < 2\sqrt{3}$	$x > 2\sqrt{3}$
Sign of $f'(x) = \frac{x^2(x+2\sqrt{3})(x-2\sqrt{3})}{(x^2-4)^2}$	$+$	$-$	$-$	$+$
$x^2(x+2\sqrt{3})(x-2\sqrt{3})$	$(+)(-)(-)$	$(+)(+)(-)$	$(+)(+)(-)$	$(+)(+)(+)$

Note: $x = -2, 2$ are not in the domain.

- Increasing interval: $(-\infty, -2\sqrt{3})$ and $(2\sqrt{3}, \infty)$
- Increasing interval: $\{(-2\sqrt{3}, 2\sqrt{3})\} - \{-2, 2\} = (-2\sqrt{3}, -2) \cup (-2, 2) \cup (2, 2\sqrt{3})$.

Therefore, from the *First Derivative Test*,
- relative maximum occurs at $x = -2\sqrt{3}$ ("+" changes to "−" at $x = -2\sqrt{3}$)
and $f(-2\sqrt{3}) = \frac{(-2\sqrt{3})^3}{(-2\sqrt{3})^2-4} = \frac{-24\sqrt{3}}{8} = -3\sqrt{3} \approx -5.1962$ is the relative maximum
- relative minimum occurs at $x = 2\sqrt{3}$ ("−" changes to "+" at $x = -\sqrt{3}$) and
$f(2\sqrt{3}) = \frac{(2\sqrt{3})^3}{(2\sqrt{3})^2-4} = \frac{24\sqrt{3}}{8} = 3\sqrt{3} \approx 5.1962$ is the relative minimum.

(g) Determine the intervals on which f is concave up and concave down. Find the points of inflection (if any).

Consider the given second derivative

$$f''(x) = \frac{8x^3 + 96x}{(x^2-4)^3}.$$

So $f''(x) = 0$ when

$$\frac{8x^3 + 96x}{(x^2-4)^3} = \frac{8x(x^2+12)}{(x+2)^3(x-2)^3} = 0 \qquad \text{or } x = 0$$

and $f''(x)$ is undefined when $x = -2, 2$. Note that $x^2 + 12 > 0$ for all $x \in \mathbb{R}$. To solve the inequalities , we will also consider the denominator (because $(x^2-4)^3$ changes sign for $x < -2$ and $x > -2$, and for $x < 2$ and $x > 2$). I.e. we will use $x = -2, 0, 2$ to subdevide the intervals.

Interval of x	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
Sign of $f''(x) = \frac{8x^3+96x}{(x^2-4)^3}$	$-$	$+$	$-$	$+$
$\frac{8(x)(x^2+12)}{(x+2)^3(x-2)^3}$	$\frac{(-)(+)}{(-)(-)}$	$\frac{(-)(+)}{(+)(-)}$	$\frac{(+)(+)}{(+)(-)}$	$\frac{(+)(+)}{(+)(+)}$

- Concave up: $f'' > 0$ when $x \in (-2, 0) \cup (2, \infty)$.
- Concave down: $f'' < 0$ when $x \in (-\infty, -2) \cup (0, 2)$.

The point of inflection is $(0, 0)$, since the curve changes the concavity at these points. Note that -2 and 2 will not give inflection points because they are not in the domain.

(h) Sketch the curve of f.

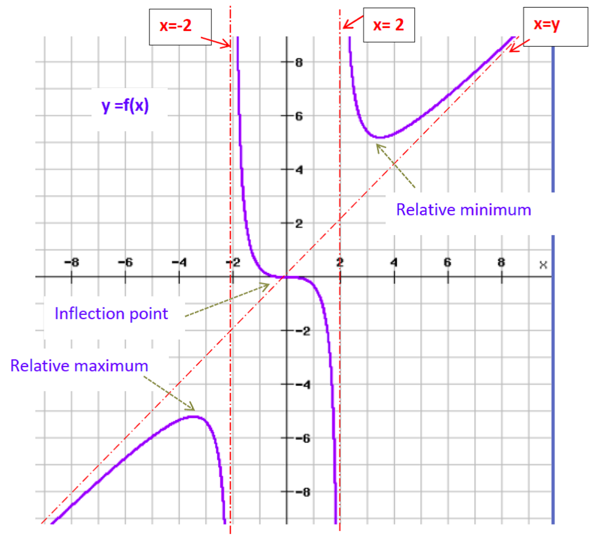

Figure 1: $f = \{(x, y)|y = \frac{x^3}{x^2-4}\}$

Optional Problems

1. Find the critical numbers of the given functions on $(-\infty, \infty)$.

   (a) $f(x) = x^3 - 3x^2 + 3x - 1$

   (b) $f(x) = \frac{x^2}{x^2+2}$

   (c) $f(x) = e^{-x} + 2x$

   (d) $f(x) = -x + \sin(x)$

   (e) $f(x) = x^2 - 8\ln(x)$

   (a) $f(x) = x^3 - 3x^2 + 3x - 1$

   Solution: $f'(x) = 3x^2 - 6x + 3$

   $$f'(x) = 0 \quad \Leftrightarrow \quad 3(x^2 - 2x + 1) = 3(x - 1)^2 = 0 \quad \Leftrightarrow \quad (x - 1) = 0 \quad \Leftrightarrow \quad x = 1$$

   Therefore, the only critical number of $f(x)$ is $x = 1$.

   (b) $f(x) = \frac{x^2}{x^2+2}$

   Solution:

   $$f'(x) = \frac{(x^2 + 2)(2x) - x^2(2x)}{(x^2 + 2)^2} = \frac{2x^3 + 4x - x^3}{(x^2 + 2)^2} = \frac{x^3 + 4x}{(x^2 + 2)^2}$$

   $f'(x) = 0$ when $x^3 + 4x = 0 \Longrightarrow x(x^2 + 4) \Longrightarrow x = 0$.

   Therefore, the only critical number of $f(x)$ is $x = 0$.

   (c) $f(x) = e^{-x} + 2x$

   Solution:

   $$f'(x) = e^{-x} + 2x = -e^{-x} + 2$$

   $f'(x) = 0$ when $-e^{-x} + 2 = 0 \Longrightarrow e^{-x} = 2 \Longrightarrow e^x = 1/2 \Longrightarrow x = \ln(1/2)$ or $x = -\ln(2)$.

   Therefore, the only critical number of $f(x)$ is $x = -\ln(2)$.

(d) $f(x) = -x + \sin(x)$
Solution:

$$f'(x) = -1 + \cos(x)$$

$f'(x) = 0$ when $-1 + \cos(x) = 0 \Longrightarrow \cos(x) = 1 \Longrightarrow x = 2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$.
Therefore, the critical numbers of $f(x)$ are $x = 2\pi n$, where $n = 0, \pm 1, \pm 2, \ldots$.

(e) $f(x) = x^2 - 8\ln(x)$
Solution:

$$f'(x) = 2x - \frac{8}{x}$$

$f'(x) = 0$ when $2x - \frac{8}{x} = 0 \Longrightarrow x = \frac{4}{x} \Longrightarrow x^2 = 4 \Longrightarrow x = \pm 2$.
Therefore, the critical numbers of $f(x)$ are $x = -2$ and $x = 2$.

2. Determine the intervals on which the given function f is increasing and the interval on which f is decreasing.

(a) $f(x) = x^2 + 6x - 1$ (b) $f(x) = x^4 - 4x^3 + 9$ (c) $f(x) = x^2 e^{-x}$

Solution: The fuction $f(x)$ is increasing when $f'(x) > 0$ and $f(x)$ is decreasing when $f'(x) < 0$.

(a) $f(x) = x^2 + 6x - 1 \Longrightarrow f'(x) = 2x + 6$

$$f'(x) = 2x + 6 > 0 \quad \Longrightarrow \quad x > -3$$

so $f(x)$ is increasing on the interval $[-3, \infty)$, and

$$f'(x) = 2x + 6 < 0 \quad \Longrightarrow \quad x < -3$$

so $f(x)$ is decreasing on the interval $(\infty, -3]$.

(b) $f(x) = x^4 - 4x^3 + 9 \Longrightarrow f'(x) = 4x^3 - 12x^2$

$$f'(x) = 4x^3 - 12x^2 > 0 \quad \Longrightarrow 4x^2(x-3) > 0 \quad \Longrightarrow x > 3$$

so $f(x)$ is increasing on the interval $[3, \infty)$, and

$$f'(x) = 4x^3 - 12x^2 < 0 \quad \Longrightarrow 4x^2(x-3) < 0 \quad \Longrightarrow x < 3$$

so $f(x)$ is decreasing on the interval $(\infty, 3]$.

(c) $f(x) = x^2 e^{-x} \Longrightarrow f'(x) = -x^2 e^{-x} + 2xe^{-x} \Longrightarrow f'(x) = xe^{-x}(2 - x)$,
since $e^{-x} > 0$, $f'(x) = 0 \Longrightarrow x = 0$ or $x = 2$:

	$x < 0$	$0 < x < 2$	$2 < x$
Sign of $f'(x) = xe^{-x}(2-x)$	$-$	$+$	$-$
$(x)(e^{-x})(2-x)$	$(-)(+)(+)$	$(+)(+)(+)$	$(+)(+)(-)$

so $f(x)$ is increasing ($f'(x) = +$) on the interval $[0, 2]$,
and $f(x)$ is decreasing ($f'(x) = -$) on the interval $(\infty, 0] \cup [2, \infty)$.

3. Use the *First Derivative Test* to find the relative extrema of the given function.

(a) $f(x) = x^3 - 3x$ (b) $f(x) = x^3 + x - 3$ (c) $\frac{x^2+3}{x+1}$ (d) $f(x) = x^2 - 2|x|$

Solution:

(a) $f(x) = x^3 - 3x \Longrightarrow f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$,
$f'(x) = 0 \Longrightarrow x = \pm 1$ are critical numbers:

	$x < -1$	$-1 < x < 1$	$1 < x$
Sign of $f'(x) = 3(x-1)(x+1)$	$+$	$-$	$+$
$(x-1)(x+1)$	$(-)(-)$	$(+)(-)$	$(+)(+)$

Therefore, from the *First Derivative Test*,
- relative maximum occurs at $x = -1$ ("$+$" changes to "$-$" at $x = -1$) and $f(-1) = (-1)^3 - 3(-1) = 2$ is the relative maximum
- relative minimum occurs at $x = 1$ ("$-$" changes to "$+$" at $x = 1$) and $f(1) = 1^3 - 3(1) = -2$ is the relative minimum.

(b) $f(x) = x^3 + x - 3 \Longrightarrow f'(x) = 3x^2 + 1 \neq 0, \forall x \in (-\infty, \infty)$.

$f'x)$ is well-defined for any x and $f'(x) \neq 0$ imply that there is *no critical number* for $f(x)$. Since a relative extremum can only occur at a critical number and there is no critical number, therefore, we conclude that **there is no relative extrema** for this function.

(c) $f(x) = \frac{x^2+3}{x+1}$ Note that the domain of this function is all real number except for $x = -1$: $D = \mathbb{R}/\{-1\}$
$\Longrightarrow f'(x) = \frac{(x+1)(2x)-(x^2+3)}{(x+1)^2} = \frac{x^2+2x-3}{(x+1)^2} = \frac{(x+3)(x-1)}{(x+1)^2}$,
$f'(x) = 0 \Longrightarrow (x+3)(x-1) = 0 \quad \Longrightarrow x = -3, 1$ are the critical numbers.

	$x < -3$	$-3 < x < 1(x \neq -1)$	$1 < x$
Sign of $f'(x)$	$+$	$-$	$+$
$(x+3)(x-1)$	$(-)(-)$	$(+)(-)$	$(+)(+)$

Therefore, from the *First Derivative Test*,
- relative maximum occurs at $x = -3$ ("$+$" changes to "$-$" at $x = -3$) and $f(-3) = \frac{(-3)^2+3}{-3+1} = -6$ is the relative maximum
- relative minimum occurs at $x = 1$ ("$-$" changes to "$+$" at $x = 1$) and $f(1) = \frac{1^2+3}{1+1} = 2$ is the relative minimum.

Notice that it is possible to have a " relative" minimum *larger* than a " relative" maximum.

(d) Note that we can write $f(x) = x^2 - 2|x|$ as:
- For $x < 0$, $f(x) = x^2 + 2x$
- For $x \geq 0$, $f(x) = x^2 - 2x$.

(i) The critical numbers:
- For $x < 0$, $f'(x) = 2x + 2$
- For $x > 0$, $f'(x) = 2x - 2$,

and $f'(x) = 0$ when $2x + 2 \Rightarrow x = -1$ or $2x - 2 = 0 \Rightarrow x = 1$. Note that $f'(x)$ does not exist when $x = 0$. Hence, the critical numbers are $x = -1, 1, 0$.

(ii) Relative extrema: Relative extrema can occur only at the critical points: $x = -1, 1, 0$. So we consider the subintervals:

	$x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x$
	$f'(x) = 2x + 2$	$f'(x) = 2x + 2$	$f'(x) = 2x - 2$	$f'(x) = 2x - 2$
Sign of $f'(x)$	$-$	$+$	$-$	$+$

Therefore, from the *First Derivative Test*,
- relative maximum occurs at $x = 0$ ("+" changes to "−" at $x = 0$)
and $f(0) = 0$ is the relative maximum
- relative minima occurs at $x = -1, 1$ ("−" changes to "+" at $x = -1, 1$) and
$f(-1) = 1^2 - 2|-1| = -1$ and $f(1) = 1^2 - 2|1| = -1$ are the relative minima.
That is, -1 is the relative minimum, which occurs at $x = \pm 1$.

4. For each given function,
(i) use the Second Derivative Test, when applicable, to find the relative extrema;
(ii) find intercepts and points of inflection, when possible;
(iii) find the intervals on which it is increasing and the interval on which it is decreasing;
(iv) find the intervals on which it is concave up and the intervals on which it is concave down;
and **(v)** sketch the graph.

(a) $f(x) = x^3 + 3x^2 + 3x + 1$
(b) $f(x) = 6x^5 - 10x^3$
(c) $f(x) = \frac{x}{x^2+2}$
(d) $f(x) = 2x - x\ln(x)$

Solution:

Note that there are 2 steps
(1) Find critical numbers c_j
(2) Check sign of $f''(c_j)$:
- If $f''(c_j) < 0$, $f(c_j)$ is a relative maximum.
- If $f''(c_j) > 0$, $f(c_j)$ is a relative minimum.
(If $f''(c_j) = 0$, no conclusion – use the First Derivative Test.)

(a) $f(x) = x^3 + 3x^2 + 3x + 1$

$$f'(x) = 3x^2 + 6x + 3, \qquad f''(x) = 6x + 6.$$

(i) Critical numbers:

$f'(x) = 3x^2 + 6x + 3 = 0 \quad \Rightarrow \quad 3(x+1)^2 = 0 \quad \Rightarrow \quad x = -1$ is the only critical number.

From the Second Derivative Test,

$$f''(-1) = 6(-1) + 6 = 0$$

so we cannot conclude anything. The First derivative will be used instead,

	$x \in (-\infty, -1)$	$x \in (-1, \infty)$
Sign of $f'(x) = 3(x+1)^2$	$+$	$+$

Therefore, from the table above, the *First Derivative Test* implies that there is **no relative extremum** since the sign of $f'(x)$ is always positive.

(ii) Find intercepts and points of inflection:
- The y-intercept(x=0) is at $y = f(0) = 0^3 + 30^2 + 3(0) + 1 = 1 \Rightarrow \ y = 1$.
- The x-intercept(y=0) is at $0 = f(x) = (x+1)^3 \Rightarrow \ x = -1$.
- To find the inflection points, we first solve $f''(x) = 0$ and see the sign change of $f''(x)$:

$$f''(x) = 6x + 6 = 0 \qquad \Longrightarrow \qquad x = -1.$$

	$x \in (-\infty, -1)$	$x \in (-1, \infty)$
Sign of $f''(x) = 6(x+1)$	$-$	$+$

Since the sign of $f''(x)$ changes at the $x = -1$, it gives an inflection point. When $x = -1$, $y = f(-1) = (-1)^3 + 3(-1)^2 + 3(-1) + 1 = 0$. Hence, the inflection point is $(-1, 0)$.
(iii) Since $f'(x) = 3(x+1)^2 > 0$for all $x \neq -1$, and $f'(x) = 0$ at $x = -1$, $f(x)$ is increasing on $(-\infty, \infty)$.
(iv) Concavity:

$$f''(x) = 6x + 6 > 0 \ \Rightarrow \ x > -1 \text{ and } \quad f''(x) = 6x + 6 < 0 \ \Rightarrow \ x < -1.$$

That is, $f(x)$ is concave up on $(-1, \infty)$ and $f(x)$ is concave down on $(-\infty, -1)$.

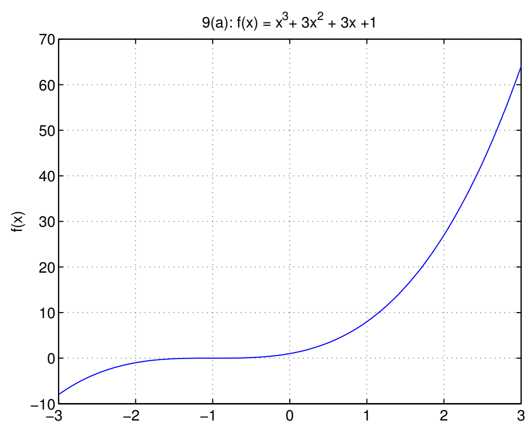

Figure 2: (a): $f(x) = x^3 + 3x^2 + 3x + 1$

(b) $f(x) = 6x^5 - 10x^3$

(i) Critical numbers:

$$f'(x) = 30x^4 - 30x^2 = 30x^2(x^2-1) = 30x^2(x-1)(x+1) = 0 \Rightarrow$$

$x = 0, -1, 1$ are the only critical numbers. From

$$f''(x) = 120x^3 - 60x = 60x(2x^2-1)$$

and from the Second Derivative Test,
$f''(0) = 0 \Longrightarrow$, no conclusion
$f''(-1) = -60 < 0 \Longrightarrow$, $f(-1) = -6 + 10 = 4$ is a relative maximum
$f''(1) = 60 > 0 \Longrightarrow$, $f(1) = 6 - 10 = -4$ is a relative minimum.
The First Derivative Test will be used to identify the critical number $x = 0$ as shown in the table. Since the sign of $f'(x)$ does not change at $x = 0$, $x = 0$ does not give an extremum.

	$x \in (-\infty, -1)$	$x \in (-1, 0)$	$x \in (0, 1)$	$x \in (1, \infty)$
Sign of $f'(x) = 30x^2(x-1)(x+1)$	$+$	$-$	$-$	$+$
$x^2(x-1)(x+1)$	$(+)(-)(-)$	$(+)(-)(+)$	$(+)(-)(+)$	$(+)(+)(+)$

(ii) Find intercepts and points of inflection:
- The y-intercept(x=0) is at $y = f(0) = 6(0)^5 - 10(0)^3 = 0 \Rightarrow \; y = 0$.
- The x-intercept(y=0) occurs when $0 = f(x) = x^3(6x^2 - 10)$
$\Rightarrow$ at $x = 0$, $x = -\sqrt{5/3}$, $x = \sqrt{5/3}$.
- To find the inflection points, we first solve $f''(x) = 0$ and see the sign change of $f''(x)$:

$$f''(x) = 60x(2x^2-1) = 0 \qquad \Longrightarrow \quad x = 0, -\sqrt{1/2}, \sqrt{1/2}.$$

	$x \in (-\infty, -\sqrt{\frac{1}{2}})$	$x \in (-\sqrt{\frac{1}{2}}, 0)$	$x \in (0, \sqrt{\frac{1}{2}})$	$x \in (\sqrt{\frac{1}{2}}, \infty)$
Sign of $f''(x) = 60x(2x^2-1)$	$-$	$+$	$-$	$+$
$(x)(x-\sqrt{1/2})(x+\sqrt{1/2})$	$(-)(-)(+)$	$(-)(-)(+)$	$(+)(-)(+)$	$(+)(+)(+)$

Since the sign of $f''(x)$ changes at the $x = 0, -\sqrt{1/2}, \sqrt{1/2}$, they give inflection points. Since $f(0) = 0$, $f(-\sqrt{1/2}) = 7\sqrt{2}/4$, $f(\sqrt{1/2}) = -7\sqrt{2}/4$,
the inflection points are $(0,0)$, $(-\sqrt{1/2}, 7\sqrt{2}/4)$,$(\sqrt{1/2}, -7\sqrt{2}/4)$.

(iii) From the table in (i), we see that
$f(x)$ is increasing ($f'(x) > 0$) on $(-\infty, -1) \cup (1, \infty)$ and
$f(x)$ is decreasing ($f'(x) < 0$) on $(-1, 1)$.

(iv) From the table in (ii), we see that
$f(x)$ is concave up ($f''(x) > 0$) on $(-\sqrt{\frac{1}{2}}, 0) \cup (\sqrt{\frac{1}{2}}, \infty)$ and
$f(x)$ is concave down ($f''(x) < 0$) on $(-\infty, -\sqrt{\frac{1}{2}}) \cup (0, \sqrt{\frac{1}{2}})$.

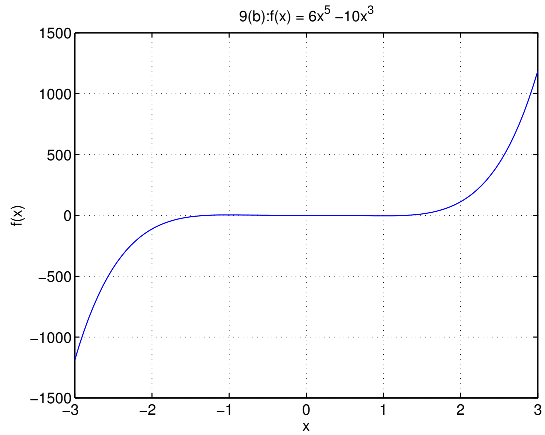

Figure 3: (b): $f(x) = 6x^5 - 10x^3$

(c) $f(x) = \frac{x}{x^2+2}$

(i) Critical numbers:

$$f'(x) = \frac{(x^2+2) - 2x^2}{(x^2+2)^2} = \frac{2-x^2}{(x^2+2)^2} = 0 \;\Rightarrow\quad 2 - x^2 = 0$$

$x = -\sqrt{2}, \sqrt{2}$ are the only critical numbers. From

$$f''(x) = \frac{2x(x^2+2)(x^2-6)}{(x^2+2)^4} = \frac{2x(x^2-6)}{(x^2+2)^3}$$

and from the Second Derivative Test,

$f''(-\sqrt{2}) = \frac{\sqrt{2}}{8} > 0 \Longrightarrow, f(-\sqrt{2}) = \frac{-\sqrt{2}}{(-\sqrt{2})^2+2} = -\sqrt{2}/4$ is a relative minimum

$f''(\sqrt{2}) = -\frac{\sqrt{2}}{8} < 0 \Longrightarrow, f(\sqrt{2}) = \frac{\sqrt{2}}{(\sqrt{2})^2+2} = \sqrt{2}/4$ is a relative maximum.

(ii) Find intercepts and points of inflection:

- The y-intercept(x=0) is at $y = f(0) = 0 \Rightarrow \; y = 0$.
- The x-intercept(y=0) occurs when $0 = f(x) \Rightarrow$ at $x = 0$.
- To find the inflection points, we first solve $f''(x) = 0$ and see the sign change of $f''(x)$:

$$f''(x) = \frac{2x(x^2-6)}{(x^2+2)^3} = 0 \qquad \Longrightarrow \quad x = 0, -\sqrt{6}, \sqrt{6}.$$

Since the denominator $(x^2+2)^3 > 0$, the sign of $f''(x)$ depends on the numerator $2x(x^2-6) = 2x(x-\sqrt{6})(x+\sqrt{6})$ as shown in the table below.

	$x \in (-\infty, -\sqrt{6})$	$x \in (-\sqrt{6}, 0)$	$x \in (0, \sqrt{6})$	$x \in (\sqrt{6}, \infty)$
Sign of $f''(x) = \frac{2x(x^2-6)}{(x^2+2)^3}$ $(2x)(x-\sqrt{6})(x+\sqrt{6})$	$-$ $(-)(-)(+)$	$+$ $(-)(-)(+)$	$-$ $(+)(-)(+)$	$+$ $(+)(+)(+)$

Since the sign of $f''(x)$ changes at the $x = 0, -\sqrt{6}, \sqrt{6}$, they give inflection points. Since $f(0) = 0$, $f(-\sqrt{6}) = \frac{-\sqrt{6}}{(-\sqrt{6})^2+2} = -\sqrt{6}/8$, $f(\sqrt{1/2}) = \frac{\sqrt{6}}{(\sqrt{6})^2+2} = \sqrt{6}/8$,
the inflection points are $(0, 0)$, $(-\sqrt{6}, -\sqrt{6}/8)$,$(\sqrt{6}, \sqrt{6}/8)$.

(iii) From

$$f'(x) = \frac{2 - x^2}{(x^2 + 2)^2} = \frac{(\sqrt{2} - x)(\sqrt{2} + x)}{(x^2 + 2)^2},$$

the denominator is always positive, so the sign of $f'(x)$ depends on the numerator: $(\sqrt{2} - x)(\sqrt{2} + x)$.

	$x \in (-\infty, -\sqrt{2})$	$x \in (-\sqrt{2}, \sqrt{2})$	$x \in (\sqrt{2}, \infty)$
Sign of $f'(x) = \frac{(\sqrt{2}-x)(\sqrt{2}+x)}{(x^2+2)^2}$	$-$	$+$	$-$
$(\sqrt{2} - x)(\sqrt{2} + x)$	$(+)(-)$	$(+)(+)$	$(-)(+)$

Hence, $f(x)$ is increasing ($f'(x) > 0$) on $(-\sqrt{2}, \sqrt{2})$ and
$f(x)$ is decreasing ($f'(x) < 0$) on $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$.

(iv) From the table in (ii), we see that
$f(x)$ is concave up ($f''(x) > 0$) on $(-\sqrt{6}, 0) \cup (\sqrt{6}, \infty)$ and
$f(x)$ is concave down ($f''(x) < 0$) on $(-\infty, -\sqrt{6}) \cup (0, \sqrt{6})$.

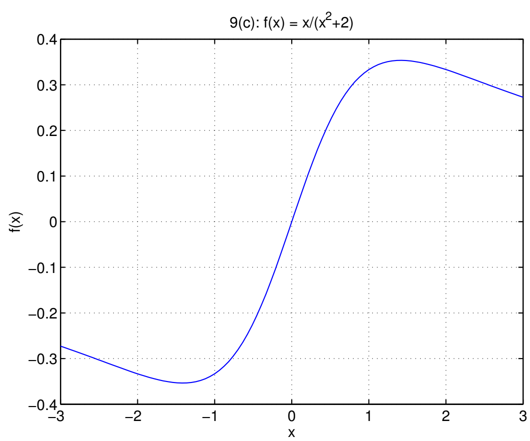

Figure 4: (c): $f(x) = \frac{x}{x^2+2}$

(d) $f(x) = 2x - x\ln(x)$
First note that the domain of f is

$$\mathcal{D} = (0, \infty),$$

since $\ln(x)$ is only well-defined for $x > 0$.
(i) Critical numbers: $f'(x) = 2 - \left[x\frac{1}{x} + \ln(x)\right] = 1 - \ln(x)$.

$$f'(x) = 1 - \ln(x) = 0 \;\Rightarrow\; \ln(x) = 1 \;\Rightarrow\; x = e^1.$$

$x = e$ is the only critical number. From

$$f''(x) = -\frac{1}{x}$$

and from the Second Derivative Test,
$f''(e) = -\frac{1}{e} < 0 \Longrightarrow$, $f(e) = 2e - e\ln(e) = 2e - e = e$ is a relative maximum.

(ii) Find intercepts and points of inflection:
- There is no y-intercept, since $x = 0$ is not in the domain of the function $f(x) = 2x - x\ln(x)$ ($\ln(x)$ is not well-defined at $x = 0$).
- The x-intercept(y=0) occurs when $0 = f(x) \Rightarrow 2x - x\ln(x) = x(2 - \ln(x)) = 0 \Rightarrow x = 0$ or $\ln(x) = 2 \Rightarrow$ at $x = 0, x = e^2$.
- To find the inflection points, we first solve $f''(x) = 0$ and see the sign change of $f''(x)$: notice that

$$f''(x) = -\frac{1}{x} \neq 0 \qquad \Longrightarrow \qquad \text{There is no inflection point.}$$

(iii) From $f'(x) = 1 - \ln(x)$,

- $f'(x) > 0$ when $1 - \ln(x) > 0 \Leftrightarrow \ln(x) < 1 \Leftrightarrow x < e \Rightarrow f(x)$ is **increasing** on $(-\infty, e]$.
- $f'(x) < 0$ when $1 - \ln(x) < 0 \Leftrightarrow \ln(x) > 1 \Leftrightarrow x > e \Rightarrow f(x)$ is **decreasing** on $[e, \infty)$.

(iv) From $f''(x) = -\frac{1}{x}$,
$f(x)$ is concave down ($f''(x) < 0 \Leftrightarrow x > 0$) on $(0, \infty)$. Note that $f(x)$ is not concave up, since $f''(x) > 0 \Leftrightarrow x < 0$, but $x < 0$ is not in the domain of $f(x)$: $\mathcal{D} = (0, \infty)$.

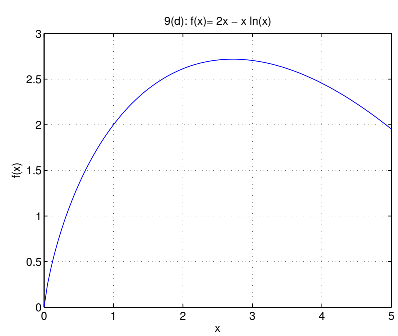

Figure 5: (d): $f(x) = 2x - x\ln(x)$

5. Let $X, Y_1, Y_2 \subseteq \mathbb{R}$. Consider the relations f_1, f_2 defined by

$$f_1 = \left\{(x,y) \in X \times Y_1 \middle| \quad y = \sqrt{\frac{x+1}{2x-3}}\right\}, \quad f_2 = \left\{(x,y) \in X \times Y_2 \middle| (x,-y) \in f_1\right\}.$$

Sketch the curve of the relation $f = f_1 \cup f_2$.

Solution:
(Details to be added...symmetry/critical numbers/increasing& decreasing/ concavity...)

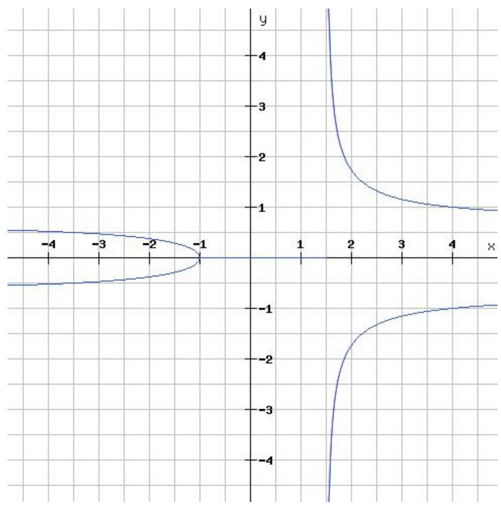

Figure 6: $f = f_1 \cup f_2 = \{(x,y) | y^2 = \frac{x+1}{2x-3}\}$

6. Find the vertical and horizontal asymptotes (if any) of $f(x) = \frac{x^2-1}{2x}$ and show that $y = \frac{x}{2}$ is a slant asymptote of f.

Solution:(Details to be added...)

- Vertical asymptote: $x = 0$
- Horizontal asymptote: None
- Slant asymptote $y = \frac{x}{2}$. We have to show that $\lim_{x\to\infty} [f(x) - \frac{x}{2}] = 0$.

$$\lim_{x\to\infty} [f(x) - \frac{x}{2}] = \lim_{x\to\infty} \left(\frac{x^2-1}{2x} - \frac{x}{2}\right) = \lim_{x\to\infty} \left(\frac{x^2-1-x^2}{2x}\right) = \lim_{x\to\infty} \frac{-1}{2x} = 0$$