4 Additional Applications of Derivative

4.1 Increasing and Decreasing Functions

The term ***increasing*** or ***decreasing*** function will be used to describe the behaviour of a function as we travel from left to right a long its graph. We can determine the behaviour of a function by comparing the value of $f(x)$ at different point x.

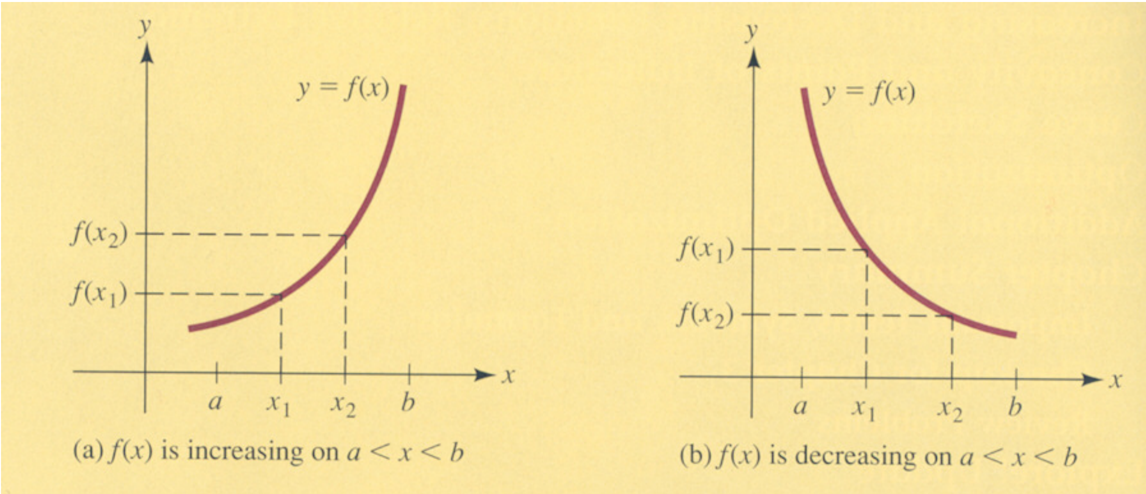


Figure 4.1 (a) Increasing and (b) decreasing function

Let $f(x)$ be a function defined on the interval $a < x < b$, and let x_1 and x_2 be two numbers in the interval. Then

$f(x)$ is **increasing** on the interval if $f(x_2) > f(x_1)$ whenever $x_2 > x_1$.

$f(x)$ is **decreasing** on the interval if $f(x_2) < f(x_1)$ whenever $x_2 > x_1$.

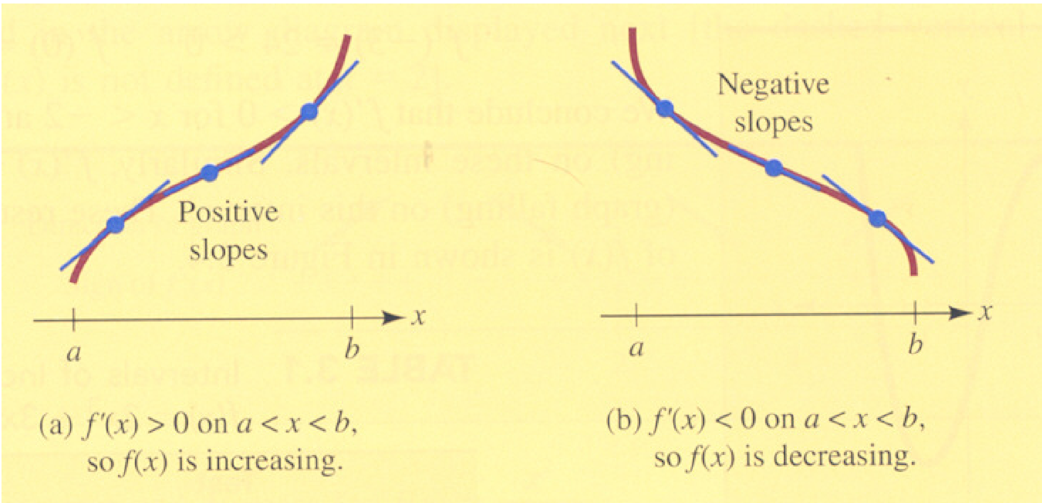


Figure 4.2: (a) Increasing and (b) decreasing function identified using derivatives.

Since the derivative $f'(x)$ of a function is the slope of its tangent, we can look at the derivative to find out which way the function changes – that is, if the function is increasing or decreasing.

4.1.1 Procedure for using the derivative to determine intervals of increasing and decreasing for a function $f(x)$

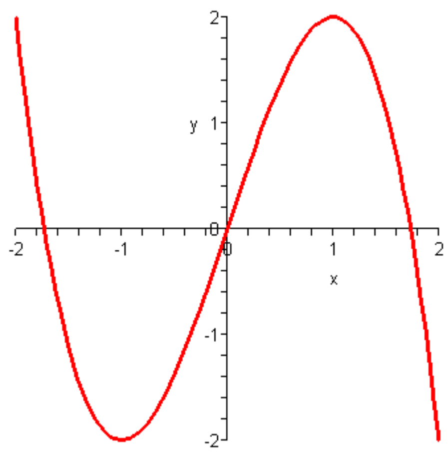
(i) Find all values of x for which $f'(x)=0$ or $f'(x)$ is not continuous, and mark these number on a number line. This divides the line into a number of open intervals.

(ii) Choose a test number c from each interval $a<x<b$ determined in (i) and evaluate $f'(c)$. Then,

- If $f'(c)>0$, the function $f(x)$ is (graph rising) on $a<x<b$.
- If $f'(c)<0$, the function $f(x)$ is (graph falling) on $a<x<b$.

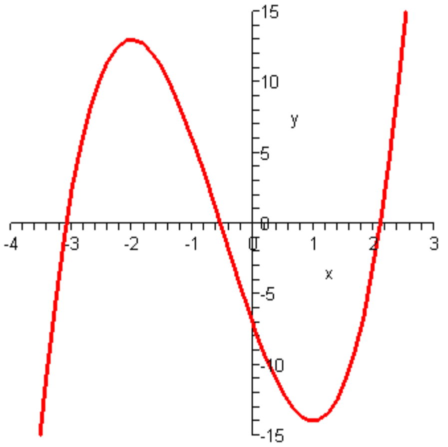
Ex. 1: Use the first derivative to find the intervals of increasing and decreasing for the function $y=x^2$.
Ans: $f(x)$ is increasing where $x>0$ and $f(x)$ is decreasing where $x<0$.

Ex. 2: Use the first derivative to find the intervals of increasing and decreasing for the function $y=3x-x^3$.



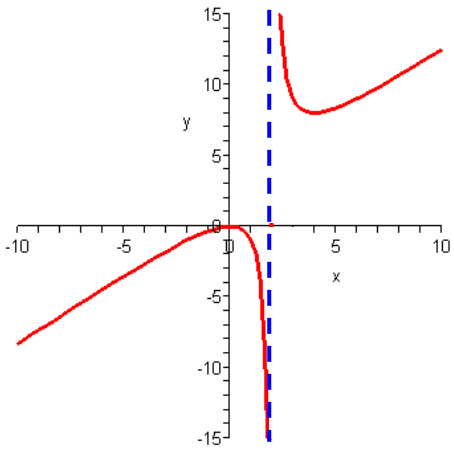
Ans: $f(x)$ is increasing for $(-1,1)$ and $f(x)$ is decreasing $(-\infty,-1)$ and $(1,\infty)$.

Ex. 3: Use the first derivative to find the intervals of increasing and decreasing for the function $y = 2x^3 + 3x^2 - 12x - 7$.



Ans: $f(x)$ is increasing for $(-\infty,-2)$ and $(1,\infty)$; and $f(x)$ is decreasing $(-2,1)$.

Ex. 4: Use the first derivative to find the intervals of increasing and decreasing for the function $y = \dfrac{x^2}{x-2}$.



Ans: $f(x)$ is increasing for $(-\infty,0)$ and $(4,\infty)$; and $f(x)$ is decreasing $(0,2)$ and $(2,4)$.

4.2 Relative Extrema

4.2.1 Introduction

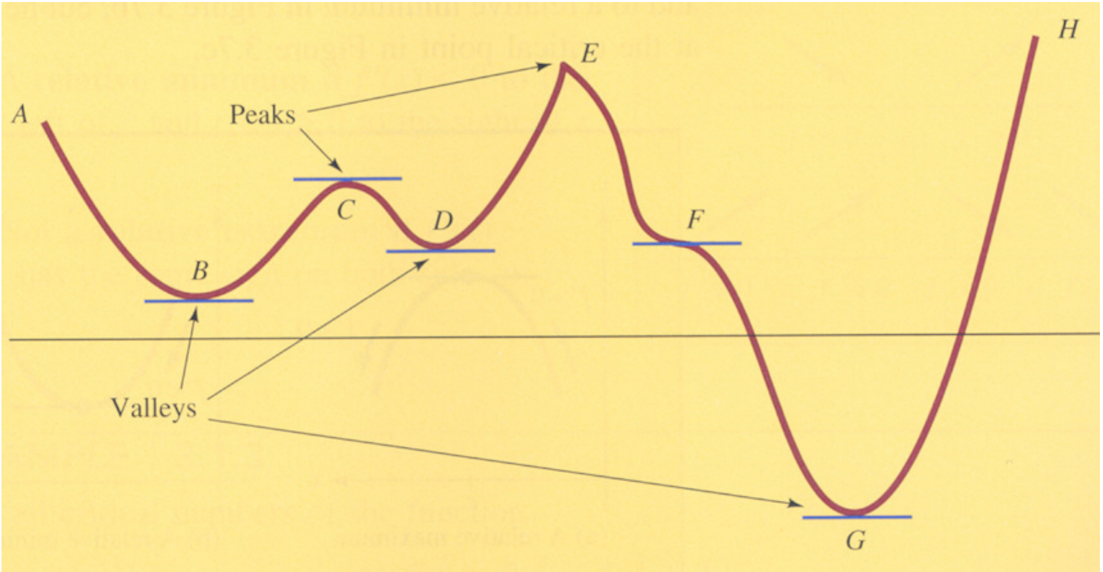


Figure 4.3 Graph of a function with various peaks and valleys

From Figure 4.3, "peaks" occur at *C* and *E* and "valleys" occur at *B*, *D* and *G*.

Points ____________________________ have horizontal tangent and

Point _________ has no unique tangent ("sharp" point with two different tangents). Hence at this point the derivative does not exist.

4.2.2 Relative Extrema

The graph of function $f(x)$ is said to have a ***relative maximum*** at $x=c$ if $f(c) \geq f(x)$ for all x in an interval $a<x<b$ containing c. Similarly, the graph has a ***relative minimum*** at $x=c$ if $f(c) \leq f(x)$ on such an interval. Collectively, the relative maxima and relative minima of f are called its ***relative extrema.***

4.2.3 Critical Number and Critical Point (Stationary point/Turning point):

A number c in the domain of $f(x)$ is called a ***critical number*** if either $\left(f'(c)=0\right)$ or $f'(c)$ does not exist. The corresponding point $\left(c, f(c)\right)$ on the graph of $f(x)$ is called a ***critical point*** for $f(x)$. ***Relative extrema can only occur at critical points.***

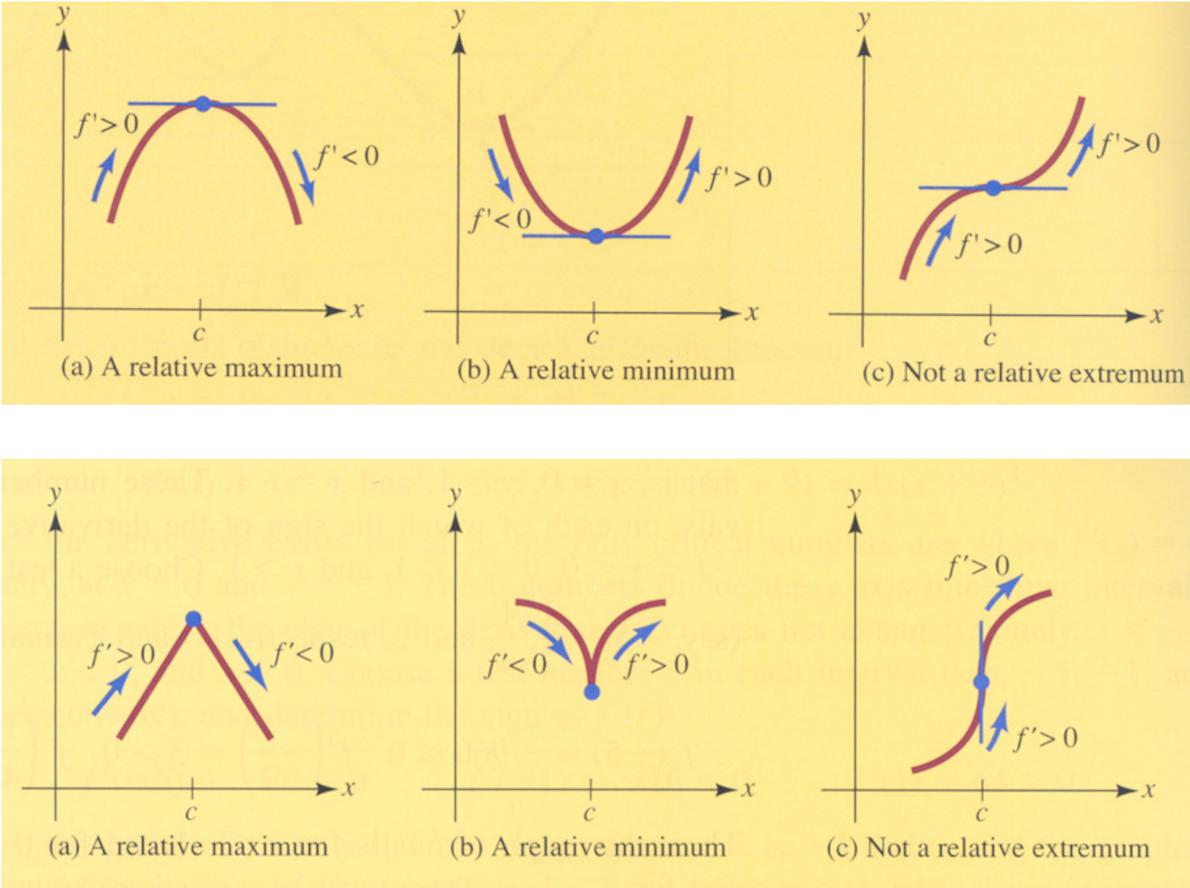
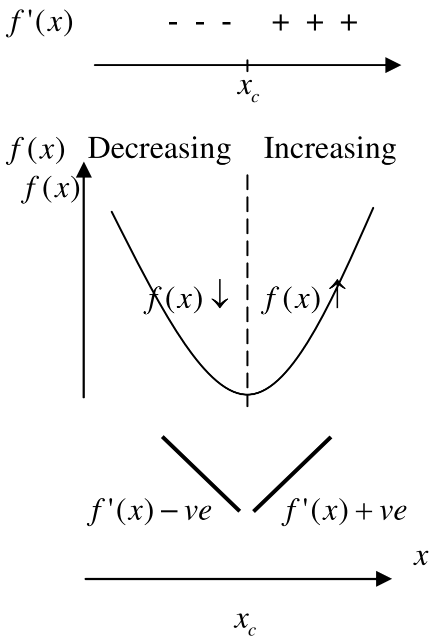
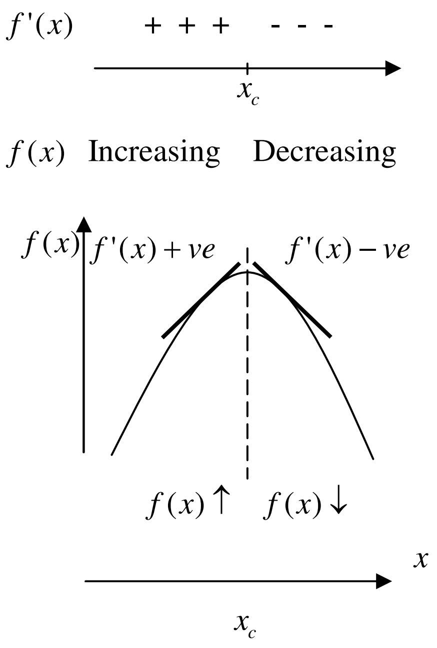


Figure 4.4: Examples of relative maximum, a relative minimum

4.2.4 First-Derivative Test for Relative Extrema

Let x_c be a critical value of f [$f(x_c)$ defined and either $f'(x_c) = 0$ or $f'(x_c)$ not defined].

Construct a sign chart for $f'(x)$ close to and on either side of x_c.

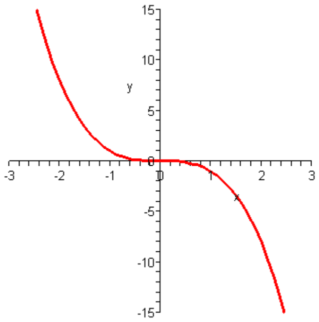
<table>
<tr><th>Sign Chart</th><th>$f(x_c)$</th></tr>
<tr><td>
</td><td>If $f'(x)$ changes from negative to positive at x_c as x is increasing, then $f(x_c)$ is a</td></tr>
<tr><td>
</td><td>If $f'(x)$ changes from positive to negative at x_c as x is increasing, then $f(x_c)$ is a</td></tr>
</table>

<table>
<tr><th>Sign Chart</th><th>$f(x_c)$</th></tr>
<tr><td>

$f'(x)$ - - -   - - -

——————+——————▶
x_c

$f(x)$ Decreasing Decreasing



</td><td>

$f(x_c)$ is not a relative extremum.

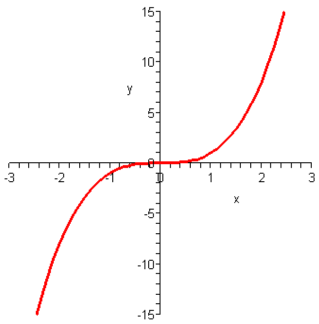
If $f'(x)$ does not change sign at x_c, then $f(x_c)$ is neither a relative maximum nor a relative minimum.

</td></tr>
<tr><td>

$f'(x)$ + + +   + + +

——————+——————▶
x_c

$f(x)$ Increasing Increasing



</td><td>

$f(x_c)$ is not a relative extremum.

If $f'(x)$ does not change sign at x_c, then $f(x_c)$ is neither a relative maximum nor a relative minimum.

</td></tr>
</table>

4.2.5 Steps of Classifying Extrema

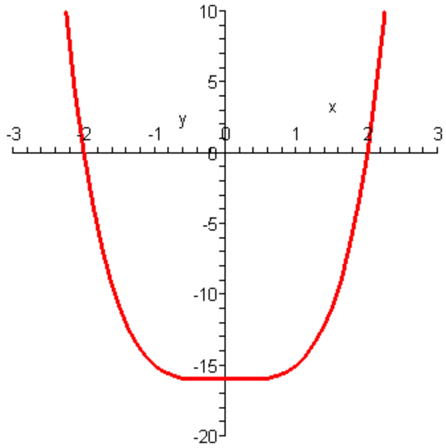
(i) Find all critical points by differentiating $f(x)$ with respect to x and equate to zero. $f'(x) = 0$.

(ii) Look at the sign of the slop on either side of the critical points and classify the type of extrema.

Ex. 5: Use the first derivative to find all critical number of the function

$$y = x^4 - 16$$

and classify each critical point as a relative maximum, a relative minimum, or neither.



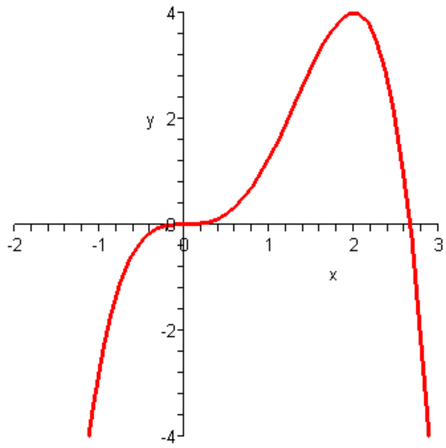
Ans: (0,-16) is a critical point and a relative minimum. $f(x)$ is increasing where $x > 0$ and $f(x)$ is decreasing where $x < 0$.

Ex. 6: Use the first derivative to find all critical number of the function

$$y = 2x^3 - \frac{3}{4}x^4$$

and classify each critical point as a relative maximum, a relative minimum, or neither.

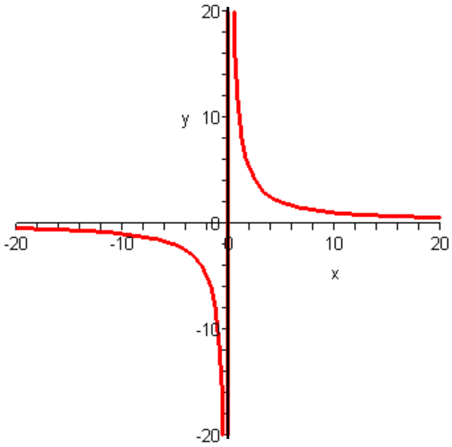
Ans: (0,0) and (2,4) are critical points and not a relative extremum and relative maxima respectively. $f(x)$ is increasing where $x < 0$ and $0 < x < 2$; and $f(x)$ is decreasing where $x > 2$.



Ex. 7: Use the first derivative to find all critical number of the function

$y = \frac{10}{x}$ for $x \neq 0$

and classify each critical point as a relative maximum, a relative minimum, or neither.



Ans: There is no critical point. Hence, there is no relative extrema. The function is increasing for all value $x \neq 0$.

4.3 Concavity (Sign of the Second Derivative) and Inflection Point

The first derivative $f'(x)$ tells us the slope of the tangent, or in other words how $f(x)$ changes as x increases. $f'(x)$ is also a function. So we can differentiate $f'(x)$, to find out how $f'(x)$ changes as x increases. We can use the second derivative to determine the shape of a function.

- The first derivative tells us if the slope is negative or positive.
- The second derivative tells us if the slope is increasing or decreasing (concavity of the curve).

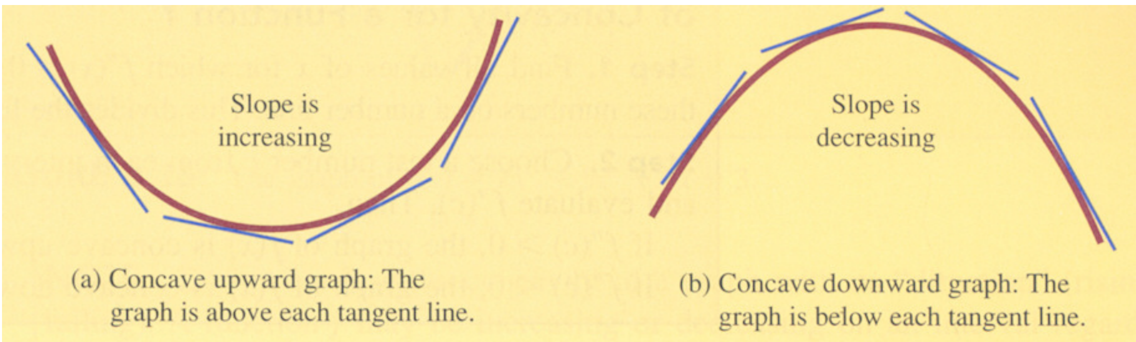


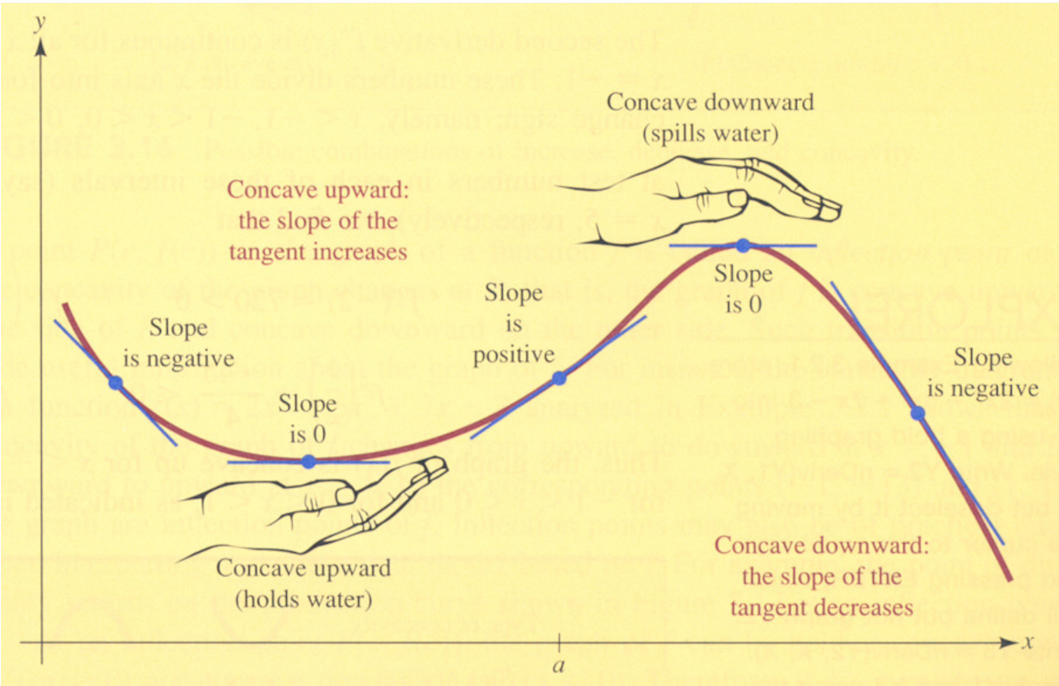
Figure 4.5: Concavity of the curve: (a) concave upward; (b) concave downward.

4.3.1 Concavity

If the function $f(x)$ is differentiable on the interval $a < x < b$, then the graph of *f* is

Concave upward on $a < x < b$ if f' is increasing on the interval

Concave downward on $a < x < b$ if f' is decreasing on the interval



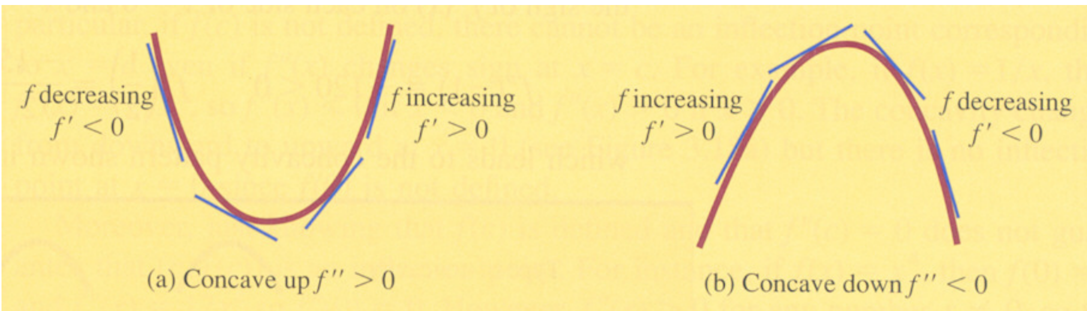


Figure 4.6 Concavity justified by second derivatives

4.3.2 Determining Interval of Concavity Using the Sign of f''

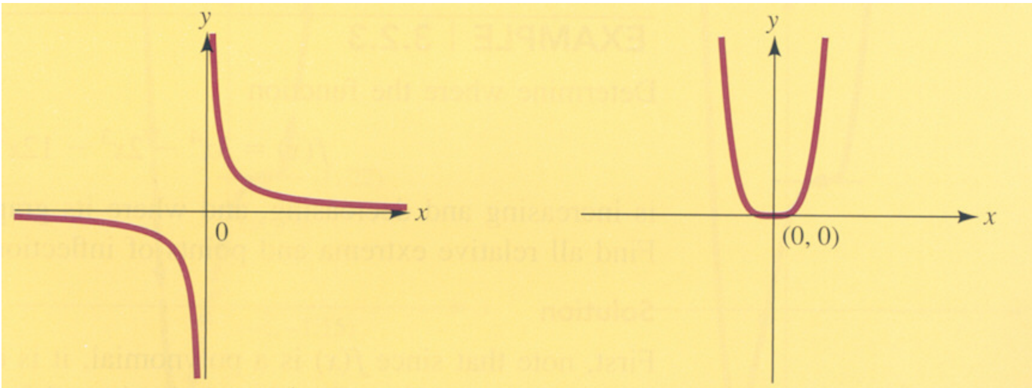
(i) Find all values of x for which $f''(x) = 0$ of $f''(x)$ does not exist, and mark these numbers on a number line. This divides the line into a number of open intervals.

(ii) Choose a test number c from each interval $a < x < b$ determined in (i) and evaluate $f''(c)$. Then,

- If $f''(c) > 0$, the graph of $f(x)$ is concave upward on $a < x < b$.
- If $f''(c) < 0$, the graph of $f(x)$ is concave downward on $a < x < b$.

4.3.3 Inflection Point

An inflection point (or point of inflection) is a point $(c, f(c))$ on the graph of a function *f* where the concavity changes. At such a point either $f''(x) = 0$ of $f''(x)$ does not exist.



(a) The graph of $f(x) = \frac{1}{x}$. The concavity changes at $x = 0$ but there is no inflection point since $f(0)$ is not defined.

(b) The graph of $f(x) = x^4$ is always concave upward, so (0, 0) is not an inflection point even though $f''(0) = 0$.

Figure 4.7: Examples of graphs where inflection does not exist.

The inflection point could also be of practical interest when interpreting a mathematical model based on *f*. For example, the point of diminishing returns on the production curve (maximum efficiency of a factory worker) in the graph below is the inflection point. The number of units that a factory worker can produce in *t* hours after arriving at work is given by a function $Q(t)$.

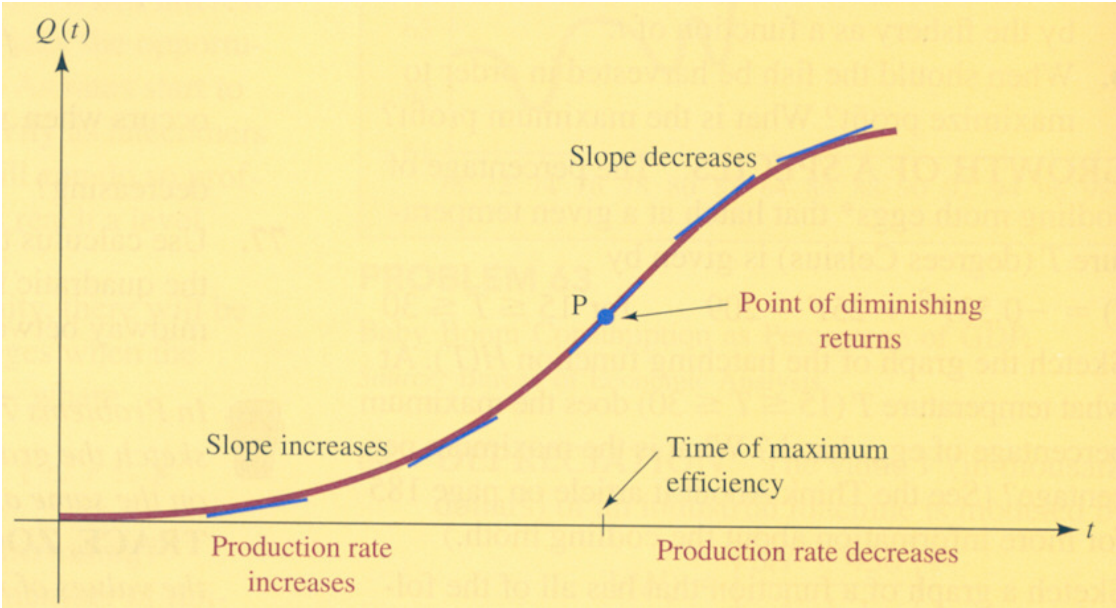
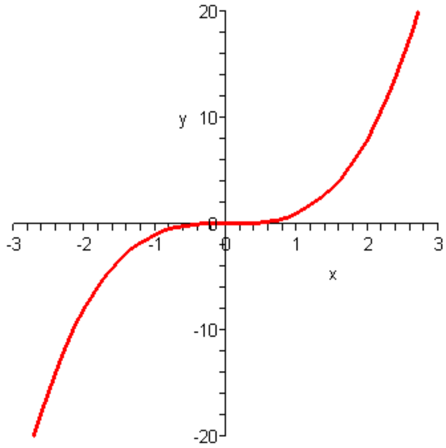
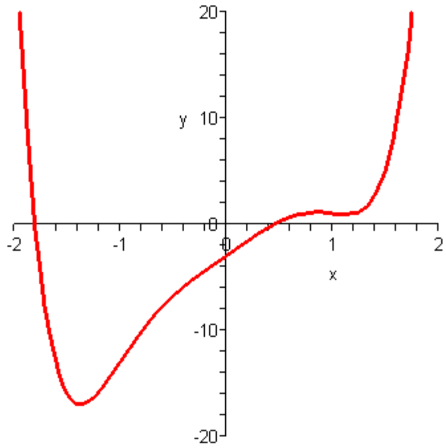


Figure 4.8: Practical interest of inflection point

Ex. 8: Determine all possible inflection points and the interval of concavity for the function $y = x^3$.



Ans: (0,0) is an inflection point. $f(x)$ is concave up where $x < 0$ and $f(x)$ is concave down where $x < 2$.



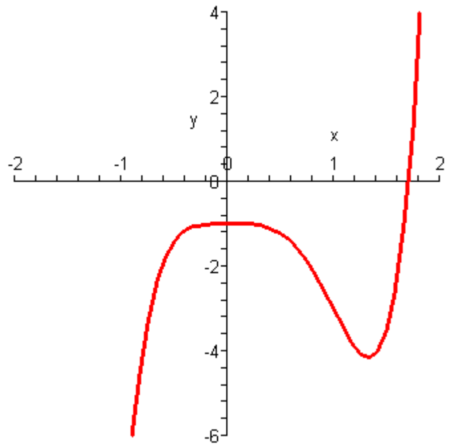
Ex. 9: Determine all possible inflection points and the interval of concavity for the function

$$y = 2x^6 - 5x^4 + 7x - 3.$$

Ans: $f''(x) = 0$ at (-1,-13), (0,-3) and (1,1). Only (-1,-13) and (1,1) are inflection points. $f(x)$ is concave up where $x < -1$ and $x > 1$; and $f(x)$ is concave down where $-1 < x < 1$.

Ex. 10: Determine all possible inflection points and the interval of concavity for the function

$f(x)=3x^5-5x^4-1$.



Ans: $f''(x)=0$ at (0,-1) and (1,-3). Only (1,-3) is an inflection point. $f(x)$ is concave up where $x>1$; and $f(x)$ is concave down where $x<1$.

4.3.4 The Second Derivative Test

Instead of using the first derivative test, we can also use the second derivative test to classify critical points.

Suppose $f''(x)$ exists on an open interval containing $x=c$ and that $f'(c)=0$.

- If $f''(c)>0$, then f has a relative at $x=c$.
- If $f''(c)<0$, then f has a relative at $x=c$.

However, if $f''(c)=0$ of $f''(c)$ does not exist, the test is inconclusive and f may have a relative maximum, a relative minimum, or no relative extremum at all at $x=c$. We need to use the first derivative test to classify the critical points.

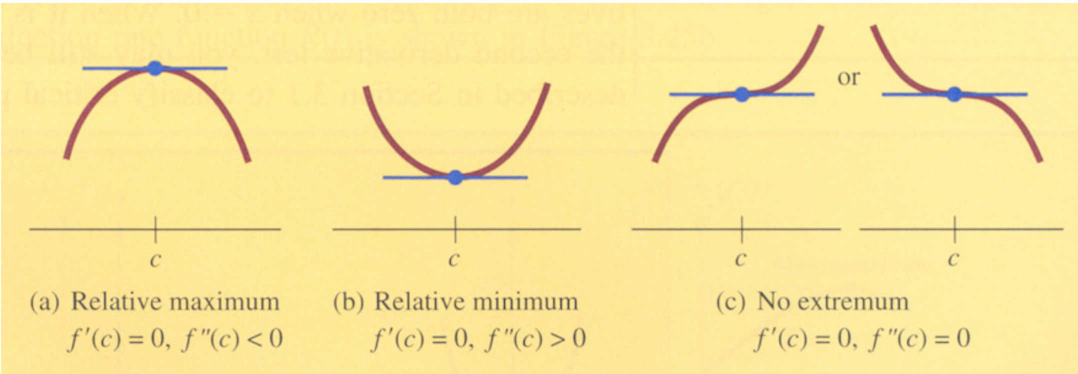


Figure 4.9 The Second Derivative Test

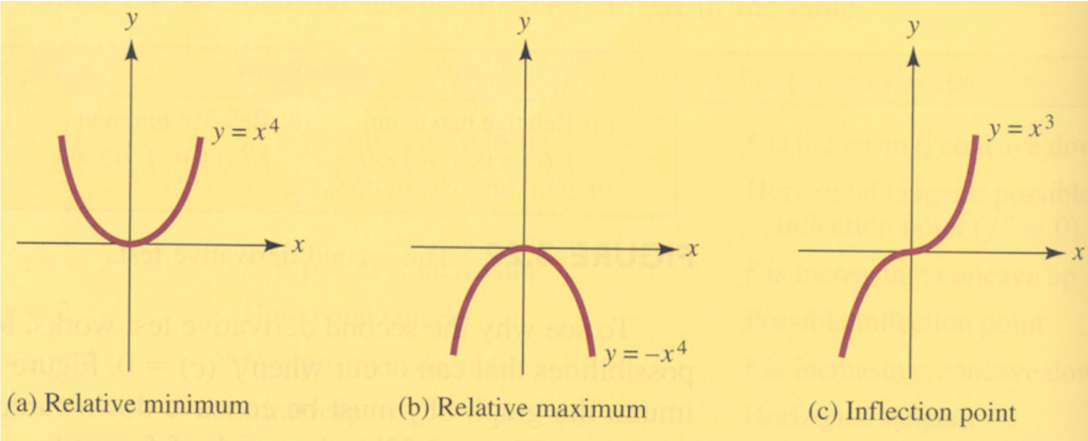
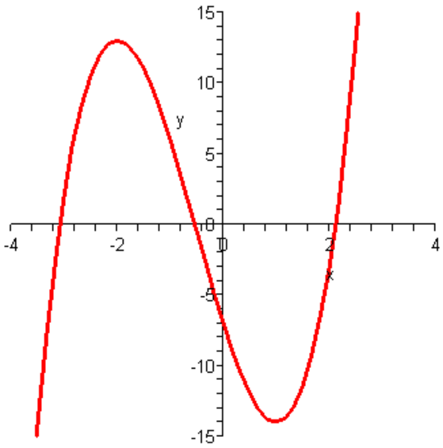


Figure 4.10 Examples of graphs with (a) relative minimum; (b) relative maximum and (c) inflection point.

Ex. 11: Find all possible critical points of $f(x) = 2x^3 + 3x^2 - 12x - 7$ and use the second derivative test to classify each critical pint as a relative maximum or minimum.

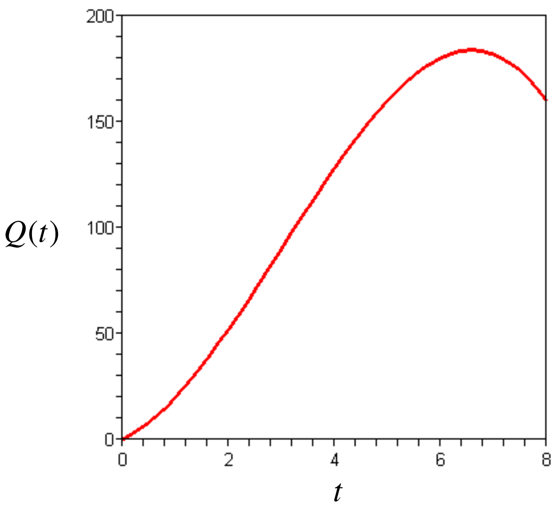
Ans: (-2,13) and (1,-14) are the critical points.

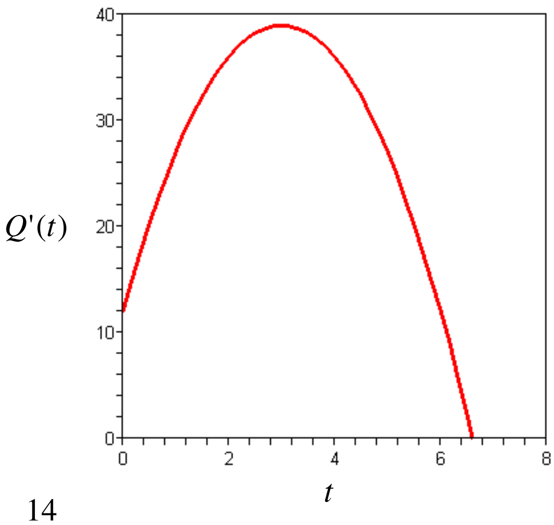
(-2,13) is a relative maximum and (1,-14) is a relative minimum.



Ex. 12: An efficiency study of the morning shift at a factory indicates that an average worker who starts at 8:00A.M. will have produced $Q(t) = -t^3 + 9t^2 + 12t$ units t hours later. At what time during the morning is the worker performing most efficiently?

Ans: 11:00A.M. $(t = 3)$



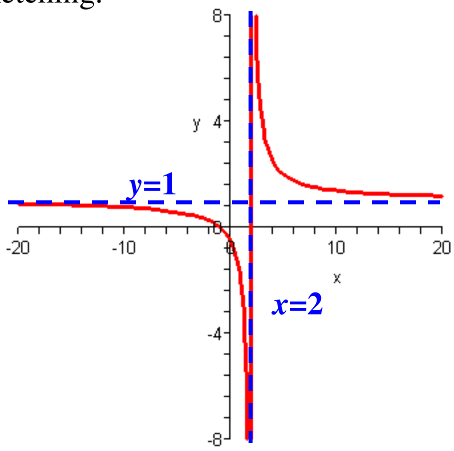


4.4 Curve Sketching

4.4.1 Asymptotes

Asymptote is a line that a curve approaches arbitrarily closely. If a function when plotting has asymptote, it will be useful to find this for curve sketching.

For example,



$$y = f(x) = \frac{x+1}{x-2}$$

$\lim_{x\to-\infty} f(x) = 1$ and $\lim_{x\to+\infty} f(x) = 1$

x	1.95	1.97	1.99	1.999	2	2.001	2.005	2.01
$f(x)$	-59	-99	-299	-2999	#DIV/0!	3001	601	301

Vertical asymptote (Any value of $x \to c$ for which $f(x)$ goes toward $\pm\infty$.)

The line $x = c$ is a ***vertical asymptote*** of the graph $f(x)$ if either

$\lim_{x\to c^-} f(x) =$ or $\lim_{x\to c^+} f(x) =$

The curve will never cross the vertical asymptote line.

For example, rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

$P(x)$ and $Q(x)$ are polynomial function. The line $x = c$ is a vertical asymptote for the graph of $f(x)$ if and only if $Q(c) = 0$ and $P(c) \neq 0$.

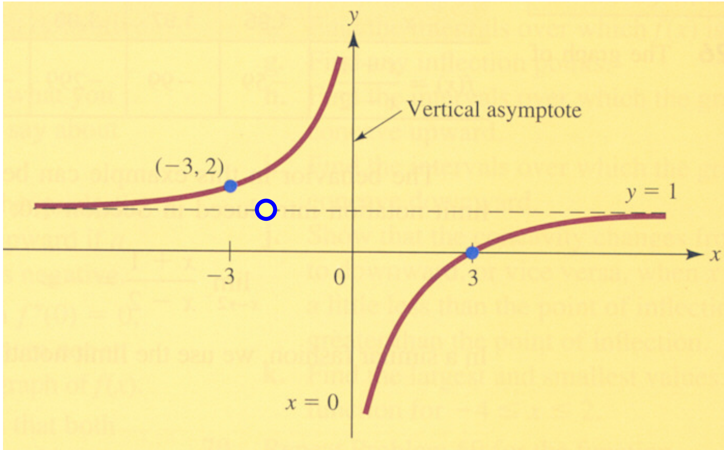
Horizontal asymptote (Value of $f(x)$ as x goes toward $\pm\infty$.)

The horizontal line $y = b$ is called a ***horizontal asymptote*** of the graph of $y = f(x)$ if

$\lim_{x\to+\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) =$

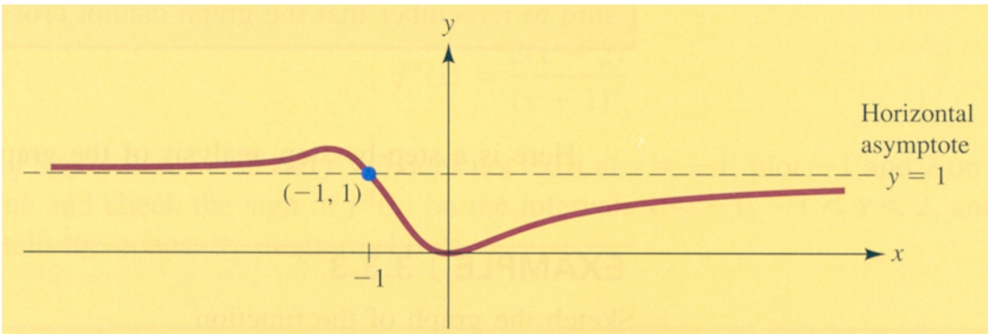
Ex. 13: Determine all asymptotes of the graph of $f(x) = \dfrac{x^2 - 9}{x^2 + 3x}$.

Ans: $x = 0$ and $y = 1$



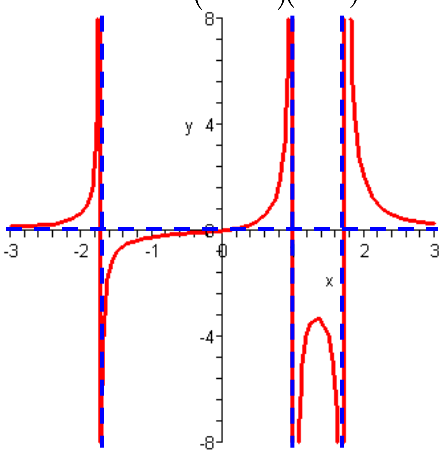
Ex. 14: Determine all asymptotes of the graph of $f(x) = \dfrac{x^2}{x^2 + x + 1}$.

Ans: $y = 1$



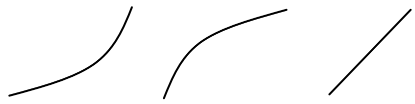
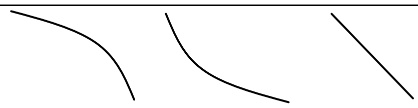
Ex. 15: Determine all asymptotes of the graph of $f(x) = \dfrac{x}{(x^2 - 4)(x - 1)}$.

Ans: $x = -2,\ x = 1,\ x = 2$ and $y = 0$



4.4.2 Information from the First-Order and the Second-Order Derivatives

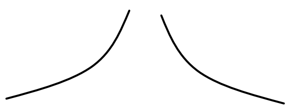
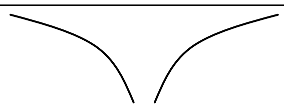
For the interval (a,b),

$f'(x)$	$f(x)$	Graph of f	Examples
+			
-			

Let x_c be a critical value of f [$f(x_c)$ defined and either $f'(x_c)=0$ or $f'(x_c)$ not defined].

$f'(x_c)$	$f''(x_c)$	Graph of f	$f(x_c)$	**Shape**
0	**+**		Relative	
0	**-**	Concave down	Relative	
0	**0**			

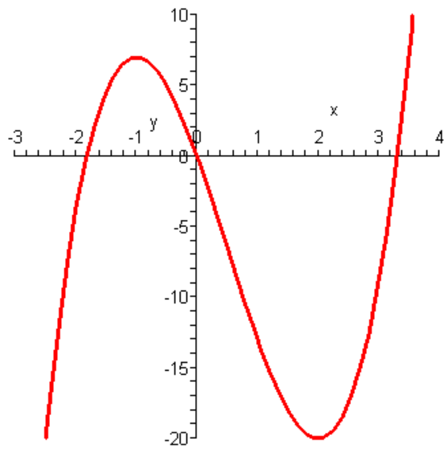
For the interval (a,b),

$f''(x)$	Graph of f	Examples
+	Concave up	
-	Concave down	

4.4.3 Guide to Curve Sketching

1) Look at **the function *f*(*x*) itself** to find the domain, *x*- or *y*- intercepts, vertical or horizontal asymptotes.
2) Look at **the first derivative** $f'(x)$ to find slop. It may be positive/negative for all values of *x*, so the function is increasing/decreasing as *x* is increasing. Otherwise, find the critical points $f'(x_c) = 0$ and check the interval where the graph is increasing (sloping up) or decreasing (sloping down).
3) Look at **the second derivative** $f''(x)$ to see the curve behaviour i.e. classify all critical points, find all inflection points and find the interval where the graph is concave up or down.
Note: Depending on the function, we may not need to do all of these steps to sketch the curve. In addition, the curve will never cross the vertical axis.

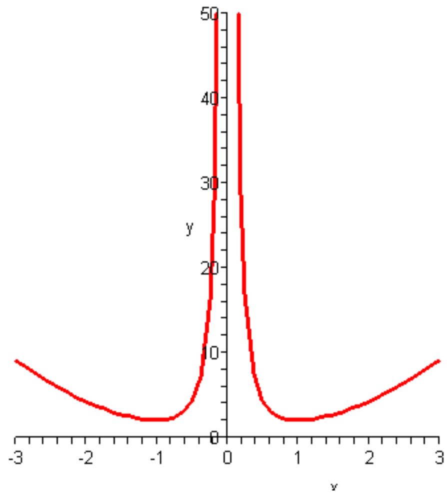
Ex. 16: Sketch the graph of the function $f(x) = 2x^3 - 3x^2 - 12x$.



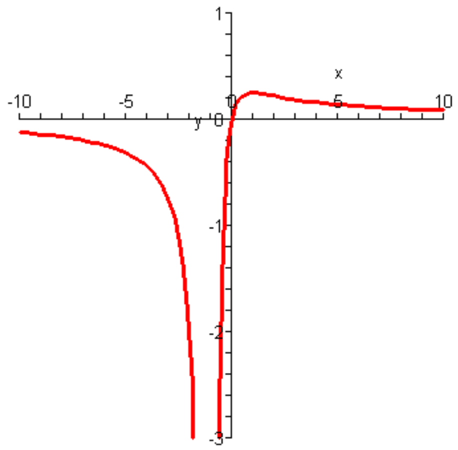
Ans: Domain = $\Re$. (0,0), (-1.81,0) and (3.31, 0) are intercepts. No asymptote. (-1,7) and (2, 20) are critical points. (-1, 7) is a relative minimum and (2, 20) is a relative maximum. $(-\infty,-1)$ and $(2,\infty)$ are increasing interval and $(-1,2)$ is decreasing interval. $(-\infty,0.5)$ is concave down interval and $(0.5,\infty)$ is concave up interval.

Ex. 17: Sketch the graph of the function $f(x) = x^2 + \frac{1}{x^2}$.

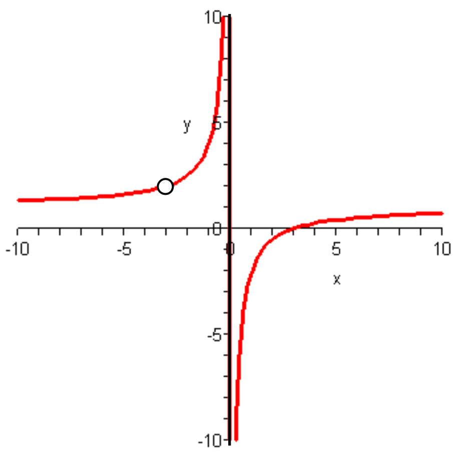
Ans: Domain = $\Re - \{0\}$. No intercept. $x = 0$ is a vertical asymptote. No horizontal asymptote. (-1,2) and (1, 2) are critical points and relative minimums. The curve is concave up for $x \neq 0$.



Ex. 18: Sketch the graph of the function $f(x)=\dfrac{x}{(x+1)^2}$.

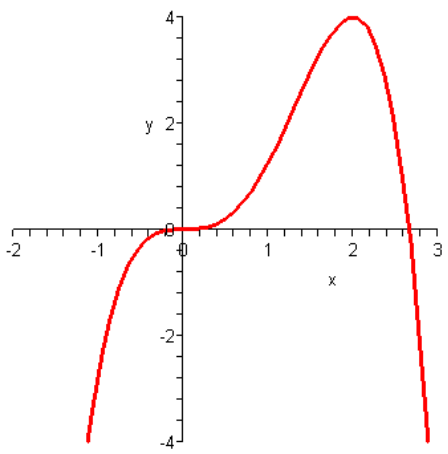


Ans: Domain = $\Re-\{-1\}$. (0,0) is the only intercept. $y=0$ is a horizontal asymptote. No vertical asymptote. $\left(1,\dfrac{1}{4}\right)$ is a critical point and a relative minimum. $\left(2,\dfrac{2}{9}\right)$ is the only inflection point. The curve is concave up for $(2,\infty)$ and concave down for $(-\infty,-1)$ and $(-1,2)$.



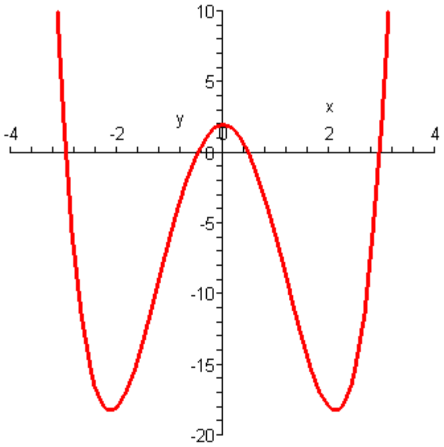
Ex. 19: Sketch the graph of the function $f(x)=\dfrac{x^2-9}{x^2+3x}$.

Ans: Domain = $\Re-\{-3\}$. (3,0) is the only intercept. $y=1$ is a horizontal asymptote and $x=0$ is a vertical asymptote.



Ex. 20: Sketch the graph of the function $f(x)=2x^3-\dfrac{3}{4}x^4$.

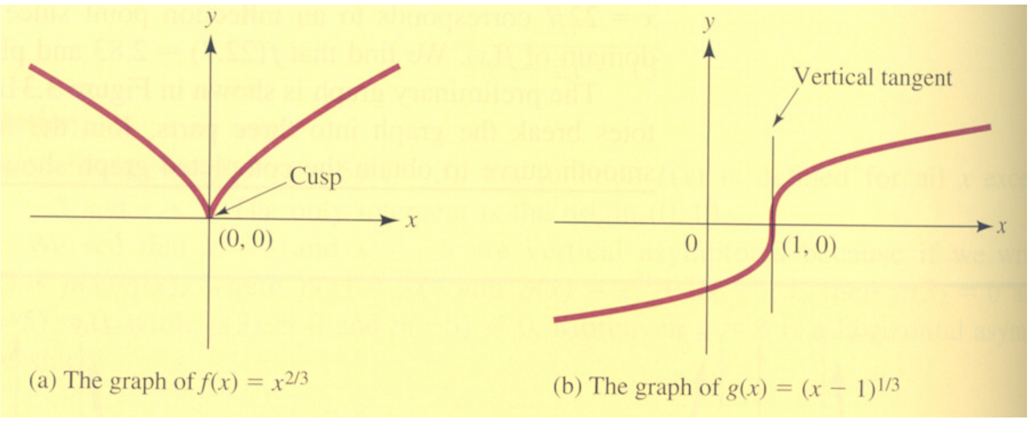
Ans: Domain = $\Re$. (0,0) and $\left(\dfrac{8}{3},0\right)$ are the intercepts. There is no asymptote. (0,0) and (2,4) are critical points. (0,0) is an inflection point and (2,4) is a relative maximum. The function is concave down where $x<0$ and $x>\dfrac{4}{3}$. The function is concave up where $0<x<\dfrac{4}{3}$.



Ex. 21: Sketch the graph of the function $f(x) = x^4 - 9x^2 + 2$.

Ans: Domain = $\Re$. (0,2), (-2.96,0), (-0.48,0), (0.48,0) and (2.96,0) are the intercepts. There is no asymptote. (0,2), (-2.12, -18.25) and (2.12,18.25) are critical points. (0,2) is a relative maximum point and (-2.12, -18.25) and (2.12,18.25) are relative minimum. The function is concave up where $x < -\sqrt{\frac{3}{2}}$ and $x > \sqrt{\frac{3}{2}}$. The function is concave down where $-\sqrt{\frac{3}{2}} < x < \sqrt{\frac{3}{2}}$.

Ex. 22: Sketch the graph of the function (a) $f(x) = x^{\frac{2}{3}}$ and (b) $g(x) = (x-1)^{\frac{1}{3}}$.



(a) The graph of $f(x) = x^{2/3}$

(b) The graph of $g(x) = (x - 1)^{1/3}$

Ex. 23: The population of a community is 230,000 in 1990 and increases at an increasing rate for 5 years, reaching the 300,000 level in 1995. It then continues to rise but at a decreasing rate until it peaks out at 350,000 in 2002. After that, the population decreases, at a decreasing rate for 3 years to 320,000 and then at an increasing rate, approaching 280,000 in the long run. Represent this information in graphical form.

4.5 Optimisation

Calculus can be used to optimise a problem; the goal is to find the absolute maximum or absolute minimum of a particular function on a relevant interval. The absolute maximum of a function on an interval is the largest value of the function on the interval and the absolute minimum is the smallest value.

If $f(x_c) \geq f(x)$ for all x in the interested domain I of f, then $f(x_c)$ is called the **absolute maximum value** of f.

If $f(x_c) \leq f(x)$ for all x in the interested domain I of f, then $f(x_c)$ is called the **absolute minimum value** of f.

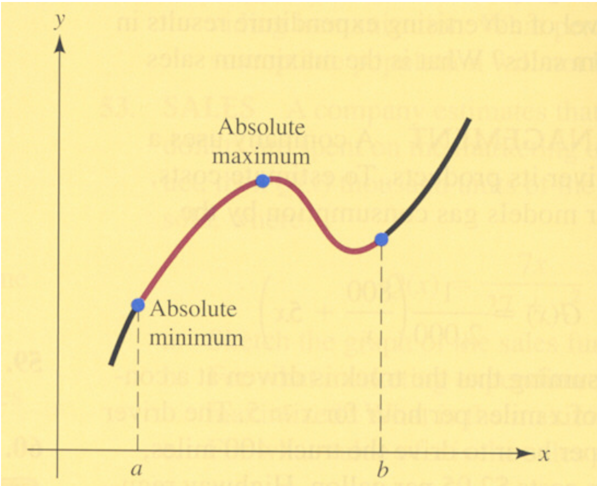
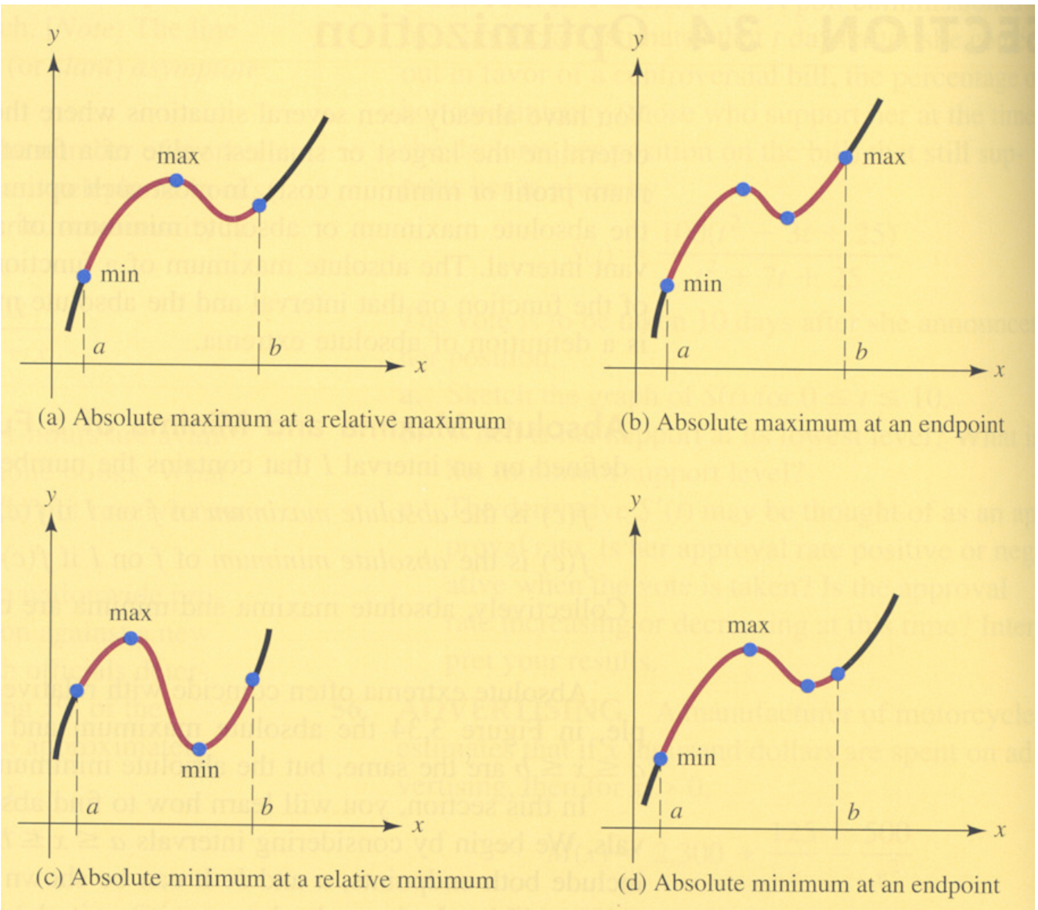


Figure 4.11: Optimisation by finding absolute minimum and absolute maxmimum

4.5.1 The Extreme Value Property

A function $f(x)$ that is continuous on the closed interval $a \leq x \leq b$ attains its absolute extrema on the interval either at a boundary point of the interval (a or b) or at a critical number such that $a < c < b$.



(a) Absolute maximum at a relative maximum
(b) Absolute maximum at an endpoint
(c) Absolute minimum at a relative minimum
(d) Absolute minimum at an endpoint

4.5.2 Procedures to fine absolute extrema on a closed interval

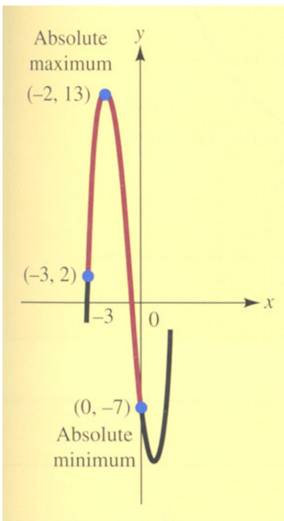
(i) Find all critical points by differentiating $f(x)$ with respect to x and equate to zero. $f'(x)=0$.

(ii) Determine the critical values $f(x_{c,i})$ **<u>in the interval</u>** (a,b).

(iii) Find boundary value $f(a)$ and $f(b)$.

(iv) Compare $f(x_{c,i})$, $f(a)$ and $f(b)$ to find absolute minimum and maximum.

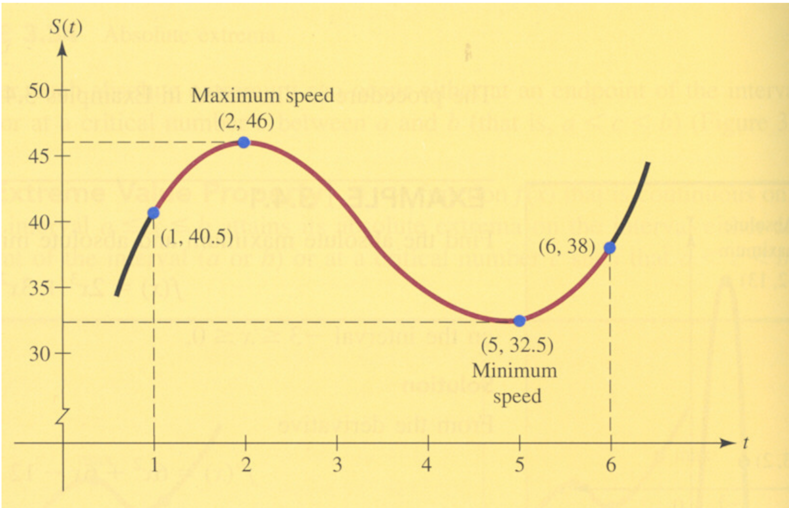
Ex. 24: Find the absolute maximum and absolute minimum of the function $f(x)=2x^3+3x^2-12x-7$ on the interval $-3 \le x \le 0$.



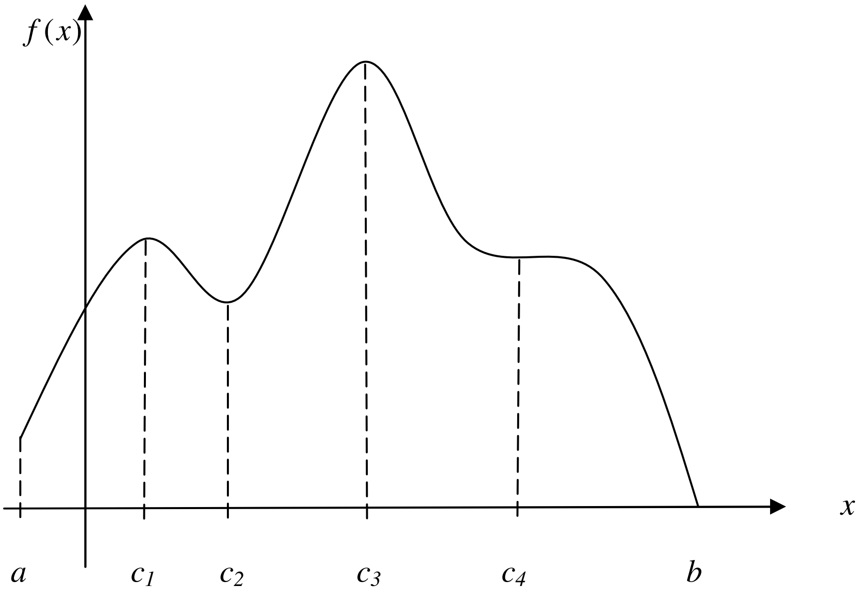
Ans: $f(-2)=13, f(-3)=2, f(0)=-7$. The absolute maximum is (-2,13) and the absolute minimum is (0,-7).

Ex. 25: For several weeks, the highway department has been recoding the speed of freeway traffic flowing past a certain downtown exit. The data suggest that between 1:00 and 6:00 P.M. on a normal weekday, the speed of the traffic at the exit is approximately $S(t) = t^3 - 10.5t^2 + 30t + 20$ miles per hour, where *t* is the number of hours past noon. At what time between 1:00 and 6:00 P.M. is the traffic moving the fastest, and at what time is it moving the slowest?

Ans: $S(1) = 40.5, S(2) = 46, S(5) = 32.5, S(6) = 38$. The traffic is moving fastest at 2:00P.M. when its speed is 46 miles per hour, and slowest at 5:00P.M., when its speeds is 32.5 miles per hour.



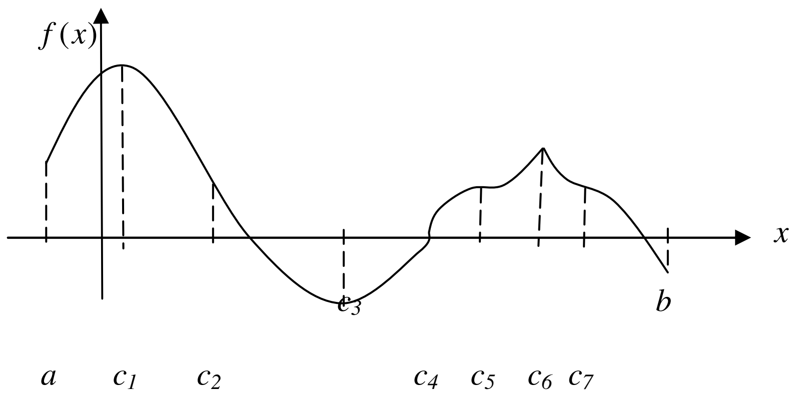
Ex. 26:



The function above has ____ critical points which are relative maximum at ____ and ____, relative minimum at ____, and critical point which neither a maximum, nor minimum at ____. This is called ______________________________. The absolute maximum of the function over the range $[a,b]$ is at ____. The absolute minimum at _____ which is not a critical point.

Note that there may not be an absolute maximum and/or minimum if there are value of x that make $f(x) = \pm\infty$.

Ex. 27: Identify the points or intervals on the x axis that produce the indicated behaviour.



$f(x)$ is increasing ______________________________

$f'(x) < 0$ ______________________________

Graph of f is concave down ______________________________

Local minima ______________________________

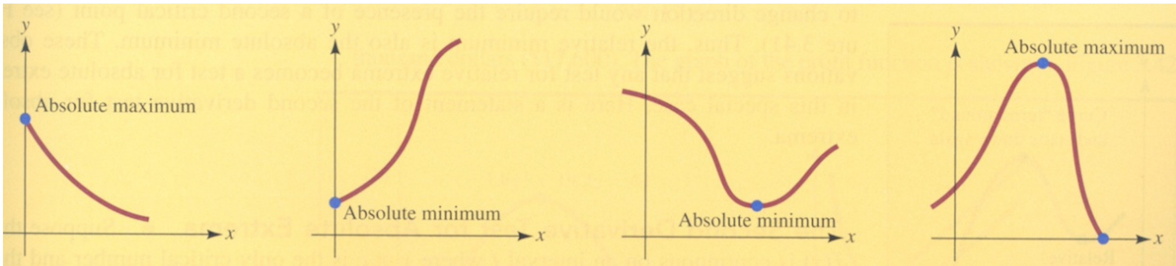
Absolute maxima ______________________________

$f'(x)$ appears to be 0 ______________________________

$f'(x)$ is not unique ______________________________

Infection points ______________________________

4.5.3 Extrema for Function Defined on Unbounded Intervals



To find the absolute extrema of a continuous function on an interval that is not of the form $a \le x \le b$, we still evaluate the function at all the critical points and boundary points that are contained in the interval. However, before we can draw any final conclusion, we must find out if the function actually has relative extrema on the interval. One way to do this is to use the first derivative to determine where the function is increasing and where it is decreasing and then to sketch the graph.

Ex. 28: If they exist, find the absolute maximum and absolute minimum of the function $f(x) = x^2 + \dfrac{16}{x}$ on the interval $x > 0$.

Ans: Absolute minimum at (2,12)

4.5.4 The Second Derivative Test for Absolute Extrema

Suppose that $f(x)$ is continuous on an interval I where $x = c$ is the **<u>only critical number</u>** and that $f'(c) = 0$. Then,

If $f''(c) > 0$, the absolute minimum of $f(x)$ on I is $f(c)$

If $f''(c) < 0$, the absolute maximum of $f(x)$ on I is $f(c)$

Ex. 29: A manufacture estimates that when q thousand units of a particular commodity are placed each month, the total cost will be $C(q) = 0.4q^2 + 3q + 40$ thousand dollars, and all q units can be sold at a price of $p(q) = 22.2 - 1.2q$ dollars per unit.

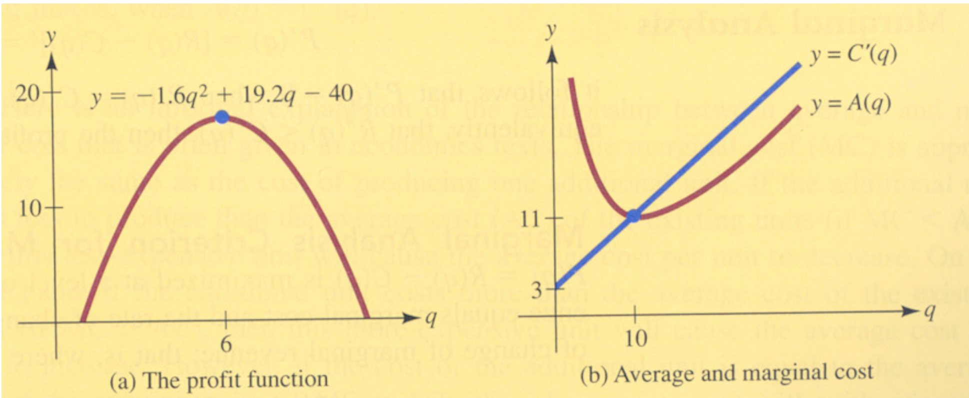
a) Determine the level of production that results in maximum profit. What is the maximum profit?

b) At what level of production is the average cost per unit $A(q) = \dfrac{C(q)}{q}$ maximised?

c) At what level of production is the average cost equal to the marginal cost $C'(q)$?

Ans: $P(q) = -1.6q^2 + 19.2q - 40$, $P(6) = 17.6$ The profit is maximised at $17,600.

$A(q) = 0.4q + 3 + \dfrac{40}{q}$, $A(10) = 11$ dollars/unit, $C'(q) = 0.8q + 3$, $q = 10$.



(a) The profit function (b) Average and marginal cost

4.5.5 Marginal Analysis Criterion for Maximum Profit

The profit $P(q) = R(q) - C(q)$ is maximised at a level of production q where marginal revenue equal marginal cost and the rage of change of marginal cost exceeds the rate of change of marginal revenue that is where

$R'(q) = C'(q)$ and $R''(q) < C''(q)$

Recall Ex. 29: The marginal revenue $R'(q) = -2.4q + 22.2$

The marginal cost $C'(q) = 0.8q + 3$

Hence,

$$R'(q) = C'(q)$$
$$-2.4q + 22.2 = 0.8q + 3$$
$$q = 6$$

and $R''(q) < C''(q)$ is also satisfied since $R''(q) = -2.4$ and $C''(q) = 0.8$

__Summary__: There are two ways to find the profit-maximising output given a cost function $C(q)$ and a demand function $p(q)$.

(1) Calculate the profit function and optimise for maximum profit.

(2) The optimal level of output for a profit maximising firm is where $\frac{dP}{dq} = 0$ or where *marginal revenue* = *marginal cost*; and $R''(q) < C''(q)$.

Marginal Analysis Criterion for Minimal Average Cost

The average cost per unit is $A(q) = \frac{C(q)}{q}$.

$$A'(q) = \frac{qC'(q) - C(q)}{q^2}$$

Average cost pre unit is minimised when

$$A'(q) = 0 = \frac{qC'(q) - C(q)}{q^2}$$
$$qC'(q) - C(q) = 0$$
$$qC'(q) = C(q)$$

or

$$C'(q) = \frac{C(q)}{q} = A(q)$$

Marginal cost = Average cost

4.5.6 Marginal Analysis Criterion for Minimal Average Cost

Average cost is minimised at the level of production where average cost equals marginal cost; that is when $A(q) = C'(q)$.

4.5.7 Price Elasticity of Demand

Price Elasticity of demand is a measure of *how 'responsive'* that demand changes for a given change in price.

$$\text{Price Elasticity of Demand, } E(p) = \frac{\text{Percentage change in demand}}{\text{Percentage change in price}}$$

$$E(p) = \frac{100 \times \dfrac{\left(\dfrac{dq}{dp}\right)}{q}}{100 \times \dfrac{\left(\dfrac{dp}{dp}\right)}{p}}$$

$$\therefore E(p) = \frac{p}{q} \cdot \frac{dq}{dp}$$

$E(p) \approx$ [percentage rate of change in demand q produced by a 1% rate of change in price q]

$E(p)$	Demand	Interpretation
$\lvert E(p) \rvert > 1$	Elastic	Demand is relatively sensitive to changes in price. A change given in price produces a larger change in demand.
$\lvert E(p) \rvert < 1$	Inelastic	Demand is relatively insensitive to changes in price. A given change in price produces a smaller change in demand.
$\lvert E(p) \rvert = 1$	Unit elastic	A given change in price produces the same change in demand.

Ex. 30: Suppose the demand q and price p for a certain commodity are related by the linear equation $q = 240 - 2p$ (for $0 \le p \le 120$)

(i) Express the elasticity of demand as a function of p.
(ii) Calculate the elasticity of demand when the price is p = 100. Interpret your answer.
(iii) Calculate the elasticity of demand when the price is p = 50. Interpret your answer.
(iv) At what price is the elasticity of demand equal to -1? What is the economic significance of this price?
(v)

Ans: $E(p) = \frac{-p}{120-p}$; $E(100) = -5$, When the price p = 100, a 1% increase in price will produce a decrease in demand of approximately 5%. The demand is elastic.; $E(50) \approx -0.71$. When the price p = 50, a 1% increase in price will produce a decrease in demand of approximately 0.71%. The demand is inelastic.; p = 60. At this price, a 1% increase in price will result in a decrease in demand of approximately the same percentage.

Ex. 31: For the demand function $q(p) = 10 - 2p$,

(i) Find the elasticity of demand [Ans: $E(p) = \frac{-2p}{10-2p}$]

(ii) Evaluate the elasticity when the price is 1 and 4 and interpret the results.

[Ans: -1/4 and -4]

4.5.8 Levels of Elasticity and the Effect on Revenue

The revenue is $R(p) = p \cdot q(p)$

Implicitly differentiate with respect to p, $\frac{dR}{dp} = p \cdot \frac{dq}{dp} + q$

Multiply the right hand side by $\frac{q}{q}$; $\frac{dR}{dp} = \frac{q}{q}\left(p \cdot \frac{dq}{dp} + q\right)$

$$\frac{dR}{dp} = q\left(\frac{p}{q} \cdot \frac{dq}{dp} + 1\right)$$

$$\frac{dR}{dp} = q[E(p) + 1]$$

If the demand is elastic, $|E(p)| > 1$, $E(p) < -1$, $[E(p) + 1] < 0$, thus

$$\frac{dR}{dp} = q[E(p) + 1] < 0$$

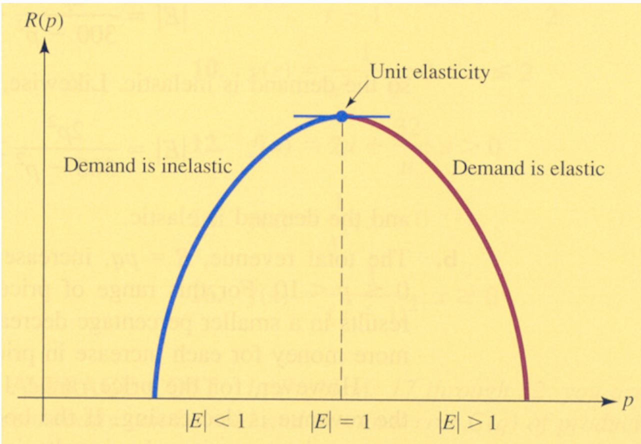
This suggests that the result of a small increase in price will be to decrease revenue.

Similarly, if the demand is inelastic, $\frac{dR}{dp} = q[E(p) + 1] > 0$. A small increase in price results in increased revenue.

If the demand is of unit elastic, then $E(p) = -1, \frac{dR}{dp} = 0$. A small increase in price leaves revenue approximately unchanged.

Summary:

- If demand is **elastic** $\left(|E(p)| > 1\right)$, revenue R decreases as price p increases.
- If demand is **inelastic** $\left(|E(p)| < 1\right)$, revenue R increases as price p increases.
- If demand is of **unity elastic** $\left(|E(p)| = 1\right)$, revenue is unaffected by a small change in price.

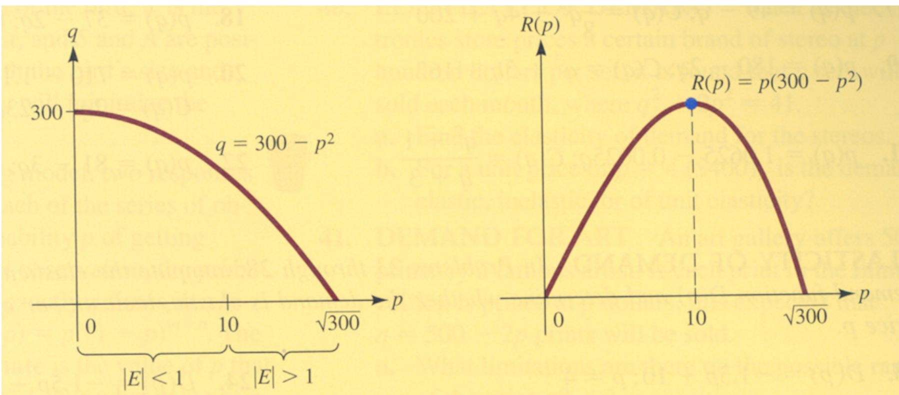


Ex. 32: The manager of a bookstore determines that when a certain new novel is priced at p dollars per copy, the daily demand will be $q = 300 - p^2$ copies, where $0 \le p < \sqrt{300}$.

(i) Determine where the demand is elastic, inelastic, and of unity elastic with respect to price.

(ii) Interpret your result in terms of the behaviour of total revenue as a function of price.

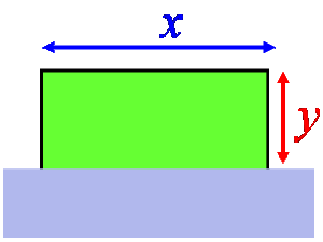
Ans: $E = \frac{-2p^2}{300 - p^2}$, $0 \le p < 10$ inelastic demand, p = 10 unity elastic, $10 < p < \sqrt{300}$ elastic.



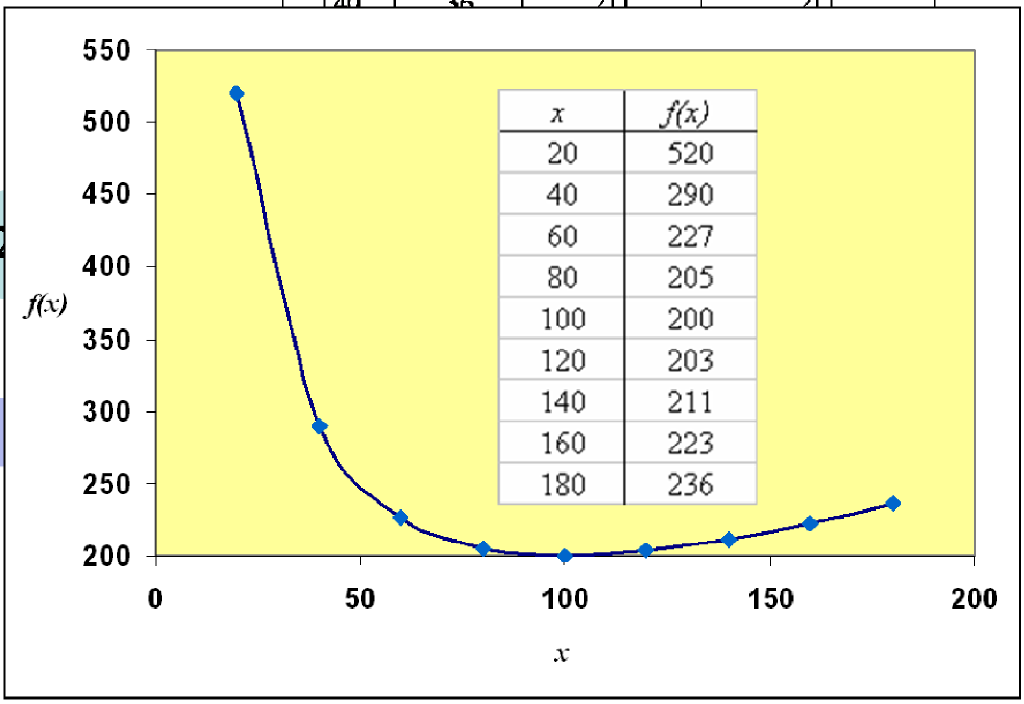
6. Additional Applied Optimisation

Ex. 33: (Recall Ex. 18 in function, graphs and limits lecture note) The highway department is planning to build a picnic area for motorist along a major highway. It is to be rectangular with an area of 5,000 square yards and is to be fenced off on the three sides not adjacent to the highway. What is the least amount of fencing requited for this job? How long and wide should the park be for the fencing to be minimised?

Ans: 200, the length parallel to the highway is 100 yards and the length perpendicular to the highway is 50 yards.



x	$y=5000/x$	$F(x,y)=x+2y$	$F(x)=x+10000/x$
20	250	520	520
40	125	290	290
60	83	227	227
80	63	205	205
100	50	200	200
120	42	203	203
140	36	211	211



Ex. 35: An office supply company sells x mechanical pencils per year at ฿p per pencil. The price demand equation for these pencils is $p = 10 - 0.001x$. What price should the company charge for these pencils to maximise their revenue? What is the maximum revenue?

Ans: $R(x) = 10x - 0.001x^2$, 5000 pencils per year at ฿5 for the maximised revenue of ฿ 25,000.

Ex. 36: The total annual cost of manufacturing x mechanical pencils for the office supply company in Ex. 35 is

$$C(x) = 5000 + 2x$$

What is the company's maximum profit? What should the company charge for each pencil and how many pencils should be produced?

Ans: $P(x) = 8x - 0.001x^2 - 5000$, 4000 pencils per year at ฿6 for the maximised profit of ฿11,000.